

UNIVERSITY OF CALIFORNIA, SAN DIEGO

Rigidity Properties of CR Embeddings into Hyperquadrics

A dissertation submitted in partial satisfaction of the
requirements for the degree
Doctor of Philosophy

in

Mathematics

by

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2011

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The dissertation of Ravi Shroff is approved, and it is acceptable in quality and form for publication on microfilm and electronically:

Chair

University of California, San Diego

2011

DEDICATION

To my parents, for their love and support.

EPIGRAPH

It's just the chain rule

—P. Ebenfelt

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ACKNOWLEDGEMENTS

I would like to thank my advisor, Peter Ebenfelt, for his help and guidance during the past five years. I am especially thankful for his endless patience with my questions and for teaching the graduate course in complex analysis that inspired me to work in his field.

I would like to thank Professors Salah Baouendi, Linda Rothschild, Mark Machina, and Joel Sobel for being on my thesis committee. I would also like to thank Professors Dmitri Zaitsev and Xiaojun Huang for their support and career advice.

I am grateful to all my friends in San Diego and the UCSD math department for helping me stay sane. Meeting and getting to know you all was the best part of graduate school. I would also like to thank my “mathematical siblings” Jiří Lebl, Jon Armel, Son Duong, and Yuan Zhang (honorary) for teaching me a large percentage of the analysis and geometry that I know, and for putting up with my many trivial questions. I must particularly acknowledge my “mathematical twin” André Minor for his friendship and help.

Finally, I would like to acknowledge my family. My parents for always encouraging me, and my brother for being a constant source of inspiration (and setting an impossibly high standard for academic excellence!). I would like to credit Joseph uncle for persuading me to study mathematics in the first place, and Georgie uncle for all his help with my undergraduate homework!

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Partial rigidity for CR embeddings of real hypersurfaces into hyperquadrics with small signature difference (with P. Ebenfelt). (*submitted*). arXiv:1011.1034v1 (2010).

ABSTRACT OF THE DISSERTATION

Rigidity Properties of CR Embeddings into Hyperquadrics

by

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Doctor of Philosophy in Mathematics

University of California, San Diego, 2011

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We study the rigidity of holomorphic mappings from a neighborhood of a Levi-nondegenerate CR hypersurface M with signature l into a hyperquadric $Q_l^N \subseteq \mathbb{C}\mathbb{P}^{N+1}$ of larger dimension and signature.

Recent work of Baouendi, Ebenfelt, and Huang shows that if the difference in signature between a source manifold with positive CR complexity and target quadric is zero then a super-rigidity phenomenon holds. Another recent paper by the same authors shows that if the difference in signature is nonnegative (and the CR complexity is zero) then a partial rigidity phenomenon occurs. This work considers both positive CR complexity and positive signature difference simultaneously, and we prove a partial rigidity result.

Our main result is that if the CR complexity of M is not too large then the image of M under any such mapping is contained in a complex plane with dimension depending only on the CR complexity and the signature difference, but not on N , the CR dimension of the target quadric. This result follows from two theorems, the first demonstrating that for sufficiently degenerate mappings, the image of M is contained in a plane, and the second relating the degeneracy of mappings into different quadrics.

Chapter 1

Introduction

The goal of this dissertation is to investigate the local rigidity properties of smooth CR mappings between embedded CR hypersurfaces in \mathbb{C}^n . By *rigidity* we essentially mean uniqueness, that is, any restriction that such a mapping must satisfy is called a rigidity phenomenon. By local, we mean that we restrict ourselves to a sufficiently small neighborhood of a point in the source manifold. We are specifically interested when the target hypersurface is a hyperquadric, when the signature difference between source and target may be positive, and when the CR complexity of the source may be positive.

The rigidity properties of mappings between hypersurfaces in complex space have been extensively investigated over many years. In section 1.1 we give an overview of the history of results regarding the rigidity of holomorphic mappings. In section 1.2 we introduce the recent history of CR embeddings into spheres and hyperquadrics. We present the theorems that motivate our work, and then state our main results. We defer the proofs of our results to chapters 4 and 5 of this dissertation.

We refer the reader to chapter 2 for relevant background, terminology, and definitions. Chapter 3 contains an introduction to and summary of some important constructions and equations from Chern-Moser and Webster's theories of pseudoconformal and pseudo-Hermitian invariants. For the reader's convenience, we end this chapter with a list of the notation that will appear in this document.

1.1 General History

To motivate the following discussion of rigidity of CR mappings, we explain briefly a relationship between CR geometry and the study of several complex variables. Recall the Riemann mapping theorem, which states that a proper open connected subset of the complex plane that is topologically equivalent to the unit disc is in fact biholomorphically equivalent to the unit disc. A fundamental question in the theory of several complex variables is to determine the extent to which a similar phenomenon holds in \mathbb{C}^n when $n > 1$. In other words, the question is to determine the biholomorphic equivalence classes of domains in \mathbb{C}^n .

It was shown by Poincaré in 1907 that the unit ball and polydisc in \mathbb{C}^2 (which are topologically equivalent) are not biholomorphically equivalent. Also, combining two celebrated results of Bochner (1943) and Fefferman (1974) yields the following:

Theorem. *Suppose D_1, D_2 are two strictly pseudoconvex domains in $\mathbb{C}^n, n > 1$, with smooth boundaries. Then the following are equivalent*

(i) *There exists a biholomorphic map $f : D_1 \rightarrow D_2$*

(ii) *There exists a smooth CR map $F : \partial D_1 \rightarrow \partial D_2$.*

This relates the classification of domains up to biholomorphic equivalence to the problem of determining when there exists a CR isomorphism between CR manifolds, which arise naturally as boundaries of domains in \mathbb{C}^n . The reader is referred to Chapter 3 of this dissertation for more information on the equivalence problem, including some aspects of the solution (by Cartan, Chern, and Moser) that will play a key role in the proofs of our results.

Perhaps the earliest well known result on rigidity of holomorphic mappings is due to Poincaré.

Theorem. *Let f be a non-constant holomorphic map from an open piece of the unit sphere $\partial\mathbb{B}^n$ into the unit sphere ($n = 2$). Then f is a linear fractional transformation and extends to a biholomorphic self-map of the unit ball. Equivalently, f is a global holomorphic automorphism of $\mathbb{C}\mathbb{P}^2$ that preserves the sphere.*

The search for higher-dimensional analogues of the preceding theorem has led to many interesting and important results, some of which we summarize here. In 1974, Alexander [A74] generalized Poincaré's result by showing that a continuous non-constant CR mapping from an open piece of the sphere in \mathbb{C}^n to itself is an automorphism of $\mathbb{C}\mathbb{P}^n$ that preserves the sphere. In 1979, Webster considered the case of non-constant CR mappings from the unit sphere in \mathbb{C}^{n+1} to the unit sphere in \mathbb{C}^{N+1} when $N = n + 1$, and showed that a similar rigidity property holds as long as the mapping is C^3 . Many further efforts concentrated on rigidity properties of mappings from the unit sphere into spheres of higher dimension (i.e. when $N - n > 1$), and with different assumptions on the regularity of the mapping. We state here a result which combines work of Ebenfelt, Forstneric, Huang, Ji, Xu, and Zaitsev:

Theorem. *Let F be a C^m -smooth ($m \geq 3$) CR map taking an open piece of the unit sphere in \mathbb{C}^n to the unit sphere in \mathbb{C}^N . If $\frac{n(n+1)}{2} + (m-2) \geq N$, then F is a rational map.*

We refer the reader to the surveys [HJ07], [Ji10] for further information on the rigidity of proper mappings between balls.

There is a close relationship between CR geometry and Riemannian geometry. On a CR manifold there is an analogue of a Riemannian metric known as the Levi form (see chapter 2, section 2), which is assumed to be nondegenerate and defined on part of the tangent space. Hence CR geometry shares features with both sub-Riemannian and pseudo-Riemannian geometry. For completeness, we state a well-known theorem of Berger, Bryant, and Griffiths, regarding rigidity of local isometric embeddings of a Riemannian manifold into Euclidean space. We refer the reader to [BBG81] and [BBG83] for further details.

Theorem. *Let $f : M^n \rightarrow \mathbb{R}^N$ be a general isometric embedding with $N = n + r$. Then (i) if $r \leq \frac{(n-1)(n-2)}{2}$, the embedding depends only on constants; (ii) if $r = \frac{(n-1)(n-2)}{2} + s$, $0 \leq s \leq n - 1$, then f depends on, at most, functions of s variables; and (iii) if*

$$\begin{aligned} r &\leq n, & n &\geq 6 \\ r &\leq 3, & n &= 4 \\ r &\leq 4, & n &= 5 \end{aligned}$$

then the embedding f is rigid.

Here rigid means determined up to an isometry of Euclidean space, and depending on constants means that the embedding is determined by its k -jet for some k .

1.2 Specific History and Outline of Results

Our motivation begins with the following result, proved by Ebenfelt, Huang, and Zaitsev, which appears in [EHZ04] as Theorem 1.2.

Theorem. *Let M be a connected smooth strictly pseudoconvex CR-hypersurface of dimension $2n + 1$. If $d < n/2$, then any smooth CR-immersion f of M into the unit sphere \mathbb{S} in \mathbb{C}^{n+d+1} is rigid. That is, any other smooth CR-immersion $\tilde{f} : M \rightarrow \mathbb{S}$ is related to f by $\tilde{f} = \phi \circ f$, where ϕ is a CR-automorphism of \mathbb{S} .*

We note that the occurrence of this rigidity depends on the dimension of the target sphere. A key ingredient in the proof of this theorem is the following result concerning degenerate mappings into the sphere. This appears in [EHZ04] as the first part of Theorem 2.2. We recall that the degeneracy of a CR transversal mapping measures the failure of the covariant derivatives of the second fundamental form of the mapping to span the whole CR normal space (see Chapter 2, [La01], or [EHZ04]).

Theorem. *Let $f : M \rightarrow \mathbb{S}$ be a smooth CR-immersion of a smooth connected CR hypersurface M of dimension $2n + 1$ into the unit sphere \mathbb{S} in \mathbb{C}^{n+d+1} . Let s be the degeneracy of f . If $d - s < n$, then $f(M)$ is contained in the intersection of \mathbb{S} with a complex plane $P \subseteq \mathbb{C}^{n+d+1}$ of codimension s .*

We call a result of this form a *partial rigidity result* because it provides a weaker geometric restriction on the embedding than, for instance, showing that the embedding is unique up to automorphisms of the target.

Our first result, Theorem 1.2.1 (stated below), generalizes the above Theorem 2.2 from [EHZ04] in two ways. First, we allow the source manifold M to be Levi-nondegenerate rather than strictly pseudoconvex, so the target is a hyperquadric with nonnegative signature. Second, we allow a difference in signature between the source and target manifolds. The proof makes extensive use of Chern-Moser pseudoconformal

invariants and Webster's pseudo-Hermitian invariants (see Chapter 3), and comprises Chapter 4 of this dissertation.

We denote by $Q_l^N \subseteq \mathbb{C}\mathbb{P}^{N+1}$ the hyperquadric of signature $0 \leq l \leq N/2$ given in homogeneous coordinates $[z_0 : z_1 : \dots : z_{N+1}]$ by

$$-\sum_{j=0}^l |z_j|^2 + \sum_{k=l+1}^{N+1} |z_k|^2 = 0.$$

Hence the superscript on Q_l^N is the CR dimension and the subscript is the signature. Also, notice that if $2l \leq N_0 < N$, then the standard linear embedding $L : \mathbb{C}\mathbb{P}^{N_0+1} \rightarrow \mathbb{C}\mathbb{P}^{N+1}$, given by

$$L([z_0 : \dots : z_{N_0+1}]) := [z_0 : \dots : z_{N_0+1} : 0 : \dots : 0],$$

satisfies $L(Q_l^{N_0}) \subseteq Q_l^N$.

Theorem 1.2.1. *Let $M \subset \mathbb{C}^{n+1}$ be a smooth connected Levi-nondegenerate hypersurface of signature $l \leq n/2$ and $f : M \rightarrow Q_l^N$ a smooth CR mapping that is CR-transversal to Q_l^N at $f(p)$ for $p \in M$. Assume that f is constantly (k, s) -degenerate near p for some k and s . If $N - n - s < n$, then there is an open neighborhood V of p in M such that $f(V)$ is contained in the intersection of Q_l^N with a complex plane $P \subset \mathbb{C}^{N+1}$ of codimension s .*

We now present a strong rigidity result proved in [BH05], where it appears as the first part of Theorem 1.6. First, we introduce some notation. If $0 \leq l \leq n-1$, we define the generalized Siegel upper-half space

$$\mathbb{S}_l^n := \{(z, w) = (z_1, \dots, z_n, w = u + iv) \in \mathbb{C}^{n+1} : v > -\sum_{j=1}^l |z_j|^2 + \sum_{j=l+1}^n |z_j|^2\},$$

where the first sum is zero if $l = 0$. The boundary of \mathbb{S}_l^n is the hyperquadric

$$\mathbb{H}_l^n := \{(z, w) = (z_1, \dots, z_n, w = u + iv) \in \mathbb{C}^{n+1} : v = -\sum_{j=1}^l |z_j|^2 + \sum_{j=l+1}^n |z_j|^2\}.$$

We use the notation \mathbb{H}_l^n to distinguish the quadric in \mathbb{C}^{n+1} from its closure in projective space, which will be what we work with. Let $Aut_0(\mathbb{H}_l^n)$ denote the stability group of the local biholomorphisms of \mathbb{C}^{n+1} preserving a piece of \mathbb{H}_l^n near the origin and sending the origin to itself.

Theorem. *Let M be a small neighborhood of 0 in \mathbb{H}_l^n with $0 < l < n$. Suppose that $F = (f_1, \dots, f_N, g)$ is a holomorphic map from a neighborhood U of M in \mathbb{C}^{n+1} into \mathbb{C}^{N+1} with $F(M) \subseteq \mathbb{H}_l^N$, $N \geq n$, and $F(0) = 0$. Suppose either $l \leq n/2$ or F preserves sides in the sense that $F(U \cap \mathbb{S}_l^n) \subseteq \overline{\mathbb{S}_l^N}$. Then if $\frac{\partial g}{\partial w}(0) \neq 0$, F is linear fractional. Moreover, there exists $\tau \in \text{Aut}_0(\mathbb{H}_l^n)$ such that either*

$$\tau \circ F(z_1, \dots, z_n, w) = (z_1, \dots, z_n, 0, \dots, 0, w), \quad \text{or}$$

$$\tau \circ F(z_1, \dots, z_n, w) = (z_{l+1}, \dots, z_n, z_1, \dots, z_l, 0, \dots, 0, -w),$$

and the latter can only happen when $l = n/2$.

Notice that the hypothesis that $l > 0$ is crucial in the above theorem. Also, in contrast with the first theorem stated above, from [EHZ04], we observe that there is no restriction on the dimension of the target quadric, hence a result of this form is sometimes called *super-rigidity*. To summarize, the above theorem shows that a mapping between two CR quadrics of positive signature, with no signature difference between source and target quadric, is linear up to automorphisms of the target.

The next natural question to consider is whether a rigidity property holds for mappings between quadrics if the target quadric is allowed to have strictly greater signature than the source. The following example shows that we may not expect more than partial rigidity to hold if there is a difference in signature between source and target. Let $\phi(z)$ be an arbitrary holomorphic function on $\mathbb{C}\mathbb{P}^{n+1}$, and define $f: Q_l^n \rightarrow Q_{l+1}^{n+2}$ by

$$f(z_0, \dots, z_{n+1}) = (z_0, \dots, z_l, \phi(z), \phi(z), z_{l+1}, \dots, z_{n+1}).$$

The map f cannot be written as the composition of a linear embedding with an automorphism of the target quadric, because any such automorphism is given by an invertible linear map preserving a quadratic form (see, for instance, section 1 of [CM74]).

However, the following result of Baouendi, Ebenfelt, and Huang (Theorem 1.1 in [BEH09]), shows that partial rigidity does occur, and moreover, after an automorphism of the target, the mapping may be put in a type of normal form. In the statement of the following theorem, we use the coordinates $(z, w) \in \mathbb{C}^n \times \mathbb{C}$ and $(z^*, w^*) \in \mathbb{C}^N \times \mathbb{C}$, and we write the components of the mapping F as $z^* = f(z, w)$, $w^* = g(z, w)$ (note that $\frac{\partial g}{\partial w}(0)$ is real).

Theorem. *Let F be a holomorphic map from an open connected neighborhood U of $0 \in \mathbb{C}^{n+1}$ into \mathbb{C}^{N+1} with $0 < n < N$ and $F(0) = 0$. Assume that F maps $\mathbb{H}_l^n \cap U$ into $\mathbb{H}_{l'}^N$, with $l \leq n/2$ and $l' \leq N/2$. Then, the following hold:*

(a) *If $\frac{\partial g}{\partial w}(0) > 0$, then $l \leq l'$ and $n - l \leq N - l'$. Moreover, if $l' < 2l$, then there is $\gamma \in \text{Aut}_0(\mathbb{H}_{l'}^N)$ such that*

$$\gamma \circ F(z, w) = (z_1, \dots, z_l, \Psi(z, w), z_{l+1}, \dots, z_n, \Psi(z, w), 0, \dots, 0, w),$$

where $\Psi = (\Psi_1, \dots, \Psi_{l'-l})$ is holomorphic near 0 (with the understanding that this term is not present when $l' = l$).

(b) *If $\frac{\partial g}{\partial w}(0) < 0$, then $l' \geq n - l$ and $N - l' \geq l$. Moreover, if $l' < n$, then there is $\gamma \in \text{Aut}_0(\mathbb{H}_{l'}^N)$ such that*

$$\gamma \circ F(z, w) = (z_{l+1}, \dots, z_n, \Psi(z, w), z_1, \dots, z_l, \Psi(z, w), 0, \dots, 0, -w),$$

where $\Psi = (\Psi_1, \dots, \Psi_{l'-(n-l)})$ is holomorphic near 0 (with the understanding that this term is not present when $l' = n - l$).

Another natural generalization is to consider the situation when M is only assumed to be embeddable in a quadric (of higher dimension) rather than be a quadric itself. Therefore, we introduce the notion of the *CR complexity*, $\mu(M)$, of a Levi-nondegenerate hypersurface $M \subseteq \mathbb{C}^{n+1}$ with signature l . The CR complexity of M measures the difference between the CR dimensions of M and a quadric Q of minimal dimension such that M is locally embeddable into Q with a CR transversal map.

$$\mu(M) := \min\{N_0 - n : \exists f_0 : M \rightarrow Q_l^{N_0} \text{ with } f_0 \text{ CR transversal to } Q_l^{N_0}\}. \quad (1.1)$$

In (1.1), we consider of course only smooth CR mappings. We note that if $l > 0$, then any f_0 in (1.1) extends holomorphically to a neighborhood of M by the well known extension theorem of Lewy (see [BER99]). If, for a given M , there are no maps f_0 as above for any N_0 , then we set $\mu(M) = \infty$ (a case which is of no interest in the present context).

The following theorem appears in [BEH08] as Theorem 1.1 and is a super-rigidity result concerning the situation where the source manifold may have positive

CR complexity, but there is no signature difference between the source manifold and target hyperquadric.

Theorem. *Let $M \subseteq \mathbb{C}^{n+1}$ be a connected real-analytic Levi-nondegenerate hypersurface of signature $l \leq n/2$. Moreover, if $l = n/2$, then assume that M is not locally biholomorphically equivalent to the hyperquadric $Q_{n/2}^n$ at any point of M . Suppose that there is an open connected neighborhood U of M in \mathbb{C}^{n+1} and a holomorphic mapping $f_0 : U \rightarrow \mathbb{C}\mathbb{P}^{N_0+1}$ with $f_0(M) \subseteq Q_1^{N_0}$ such that $f_0(U) \not\subseteq Q_1^{N_0}$. If $f : U \rightarrow \mathbb{C}\mathbb{P}^{N+1}$ is a holomorphic mapping with $f(M) \subseteq Q_1^N$, $f(U) \not\subseteq Q_1^N$, and $N_0 - n < l$, then there is $T \in \text{Aut}(Q_1^N)$ such that $f := T \circ L \circ f_0$, where L denotes the standard linear embedding given above.*

We now present our main result, where we allow both positive CR complexity of the source manifold and a positive difference in signature between source manifold and target hyperquadric. This is a partial rigidity result which generalizes the theorem above from [BEH09] except for producing a normal form for the holomorphic map. The proof of this result, contained in the last section of Chapter 5, follows from Theorem 1.2.1 above and Theorem 5.0.1. Theorem 5.0.1 is the main technical result of this dissertation, and it is stated and proved in Chapter 5.

Theorem 1.2.2. *Let $M \subseteq \mathbb{C}^{n+1}$ be a smooth connected Levi-nondegenerate CR hypersurface with signature $l \leq n/2$ and CR complexity $\mu := \mu(M)$. Let U be an open neighborhood of M in \mathbb{C}^{n+1} , and $f : U \rightarrow \mathbb{C}\mathbb{P}^{N+1}$ a holomorphic mapping with $f(M) \subseteq Q_1^N$ and f CR transversal to Q_1^N along M . Then the following hold*

(a) *If $l = n/2$ or f is side preserving then $l' \geq l$ and $N - l' \geq n - l$. If either*

(i) $\mu + (l' - l) < l$ or

(ii) $\mu + \min(l' - l, (N - l') - (n - l)) < n$ and $(N - l') - (n - l) < l$,

then $f(M) \subseteq Q_1^N \cap P$, where $P \subseteq \mathbb{C}^{N+1}$ is a complex plane of dimension $(n + 1) + \mu + \min(l' - l, (N - l') - (n - l))$.

(b) *If f is side reversing then $N - l' \geq l$ and $l' \geq n - l$. If $l' < n$ and $\mu + (l' + l - n) < n$ then $f(M) \subseteq Q_1^N \cap P$, where $P \subseteq \mathbb{C}^{N+1}$ is a complex plane of dimension $(n + 1) + \mu + (l' + l - n)$.*

We make several remarks.

Remark 1. In the arguably most interesting case where the codimension $N - n$ is large (say $N - n \geq n$) then the theorem simply states that the image of, say, a side preserving map is contained in a complex plane of dimension $(n + 1) + \mu + (l' - l)$ provided that $\mu + (l' - l) < l$. Moreover, if the signature difference is also so small that $l' - l < n - 2l$, then the theorem (by the first statement in (b)) also implies that any map is necessarily side preserving. Thus, a corollary of Theorem 1.2.2 is the following: *Let M and f be as in Theorem 1.2.2. If $N - n \geq n$, $l' - l < n - 2l$, and $\mu + (l' - l) < l$, then the image of f is contained in a complex plane of dimension $(n + 1) + \mu + (l' - l)$.* Here, no distinction needs to be made about the map being side preserving or reversing.

Remark 2. We observe that if $l' = N/2$ then the inequalities $(N - l') - (n - l) < l$ and $l' < n$ are equivalent and the conclusions of parts (a) and (b) of Theorem 1.2.2 coincide. We also observe that if f is side preserving, either assumption (i) or (ii) could apply. For instance, if $n = 5$, $l = 1$, $N = 7$, $l' = 3$, and $N_0 = 6$, then assumption (i) does not hold, but assumption (ii) does. However, if N is sufficiently large (i) may hold but not (ii).

1.3 Notation and an Example

Throughout this document, unless otherwise indicated we will use the *Einstein summation convention*, meaning that if an repeated index is present in both an upper and lower position in a mathematical expression, then the index is to be summed over. For example, $a_i \theta^i := a_1 \theta^1 + \dots + a_n \theta^n$.

$T_p^{0,1}M$ is the CR tangent space to M .

ω_α^β is the Tanaka Webster connection on M .

Q_l^n is the CR hyperquadric of CR dimension n and signature l .

$g_{\alpha\bar{\beta}}$ is generally the Levi form on M .

(θ, θ^A) is a coframe on M and θ will be a contact form.

$(\omega_\alpha^a \beta)$ is the second fundamental form of $f : M \rightarrow \hat{M}$.

(T, L_A) is a frame dual to (θ, θ^A) , and T will be the associated *Reeb* vector field.

E_k is the span of $\omega_\alpha^a \beta$ and its covariant derivatives up to order k .

We now present a well-known example of a map $W : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{2n+1}$ which shows that the codimension restriction in the first theorem of the previous section (Theorem 1.2 in [EHZ04]) is necessary for rigidity. The map is known as the *Whitney map* and is given by

$$W(z_1, \dots, z_{n+1}) = (z_1, \dots, z_n, z_1 z_{n+1}, z_2 z_{n+1}, \dots, z_n z_{n+1}, z_{n+1}^2).$$

It is easy to see that the Whitney map takes the unit sphere in \mathbb{C}^{n+1} to the unit sphere in \mathbb{C}^{2n+1} , and that the codimension d equals n , which is not strictly less than $n/2$. We denote the standard linear embedding of \mathbb{C}^{n+1} into \mathbb{C}^{2n+1} by L , and notice that the Whitney map cannot be written as the composition of the standard linear embedding with a CR automorphism of the sphere. This is because any CR automorphism of the sphere is a linear fractional transformation, and the Whitney map is a polynomial mapping of degree 2.

For further examples of explicit inequivalent mappings between quadrics, we refer the reader to recent work of Lebl ([Leb10]).

Chapter 2

Preliminaries

We collect some definitions and theorems that will be used in this dissertation. We assume the reader is familiar with basic real and complex differential geometry and complex analysis. For further background on the theory of several complex variables, we recommend the reader consult [BER99], [Hör90], [DA93], or [FG02]. For (real) differential geometry, we recommend [Le02], for Riemannian geometry [Le97], and for CR geometry [BER99]. For more specific references on the material in sections 2.4, 2.5, and 2.6, we recommend the reader consult the papers cited in those sections.

2.1 Basic CR geometry

We will generally work in \mathbb{C}^n with the coordinates $z = (z_1, \dots, z_n)$, where $z_j = x_j + iy_j$. We identify \mathbb{C}^n with \mathbb{R}^{2n} via $z \rightarrow (x_1, y_1, \dots, x_n, y_n)$. Complex conjugation is denoted with a bar, so $\bar{z}_j = x_j - iy_j$. The *complexified tangent space* to \mathbb{C}^n at a point p is defined to be $\mathbb{C} \otimes (T_p \mathbb{R}^{2n})$, and the bundle will be denoted by $\mathbb{C}T\mathbb{C}^n$. Note that for each $p \in \mathbb{C}^n$, $\mathbb{C}T_p\mathbb{C}^n$ is a complex vector space of complex dimension $2n$ and real dimension $4n$.

At each point, we may change basis on $\mathbb{C}T_p\mathbb{C}^n$ by defining the complex differential operators $\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j})$ and $\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j})$. These differential operators are defined at the point p , but for notational convenience we will usually suppress the dependence on the point except as necessary. We define the complex differentials $d\bar{z}^j = dx^j + idy^j$ and $d\bar{z}^j = dx^j - idy^j$ in $\mathbb{C}T_p^*\mathbb{C}^n$ in terms of the real differentials. We

may decompose $\mathbb{C}T_p\mathbb{C}^n$ as the direct sum $T_p^{1,0}\mathbb{C}^n \oplus T_p^{0,1}\mathbb{C}^n$, where $T_p^{1,0}\mathbb{C}^n = \text{span}(\frac{\partial}{\partial z_j})$ for $j = 1, \dots, n$ and $T_p^{0,1}\mathbb{C}^n = \overline{T_p^{1,0}\mathbb{C}^n}$.

Notice that $dz^j(\frac{\partial}{\partial z_k}) = d\bar{z}^j(\frac{\partial}{\partial \bar{z}_k}) = \delta_j^k$ and $dz^j(\frac{\partial}{\partial \bar{z}_k}) = d\bar{z}^j(\frac{\partial}{\partial z_k}) = 0$ for all j, k . For a C^1 function f defined in an open set in \mathbb{C}^n , we define the complex differential of f at the point $p \in U$ by

$$df(p) = \frac{\partial f}{\partial z_j}(p)dz^j + \frac{\partial f}{\partial \bar{z}_j}(p)d\bar{z}^j$$

where we are using the Einstein summation convention.

For $U \subseteq \mathbb{C}^n$, we say that a function $f : U \rightarrow \mathbb{C}$ is *real-analytic* if for each point $p \in U$ we can write f as a convergent power series in a neighborhood of p ,

$$f(z, \bar{z}) = \sum_{\alpha} c_{\alpha\beta} (z - p)^{\alpha} (\bar{z} - \bar{p})^{\beta}$$

for complex constants $c_{\alpha\beta}$, where the series converges to f . If f is real-valued then $c_{\alpha\beta} = \overline{c_{\beta\alpha}}$. Here we are using multi-index notation, so $\alpha = (\alpha_1, \dots, \alpha_n)$ where each α_j is a nonnegative integer, and $z^{\alpha} = z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}$.

If in fact for a C^1 function $f : U \rightarrow \mathbb{C}$ we can write in a neighborhood of each point p

$$f(z) = \sum_{\alpha} c_{\alpha} (z - p)^{\alpha}$$

where this power series has a positive radius of convergence and converges to f , then we say that f is (*complex*) *analytic* in U . It can be shown that this is equivalent to $\frac{\partial f}{\partial \bar{z}_j} = 0$ for each j , so we will use the term *holomorphic* interchangeably with analytic. We call a mapping $f : \mathbb{C}^n \rightarrow \mathbb{C}^N$ holomorphic if each component is holomorphic, and call a mapping biholomorphic if it has a holomorphic inverse.

We now introduce the fundamental object of study, the *CR hypersurface*. The theory of CR manifolds is vast, and we restrict ourselves to studying real codimension one submanifolds of \mathbb{C}^{n+1} (although we will sometimes embed our hypersurfaces into complex projective space). For background on higher codimension CR manifolds and the theory of abstract (not necessarily embedded in \mathbb{C}^{n+1}) CR manifolds, the reader may consult [BER99]. We also mention that all questions addressed in this dissertation are local in nature, so we always work in a sufficiently small neighborhood of a point. Note that our hypersurfaces are embedded in \mathbb{C}^{n+1} as opposed to \mathbb{C}^n for notational

convenience; as will be explained below, if M is a hypersurface in \mathbb{C}^{n+1} , it will have CR dimension n .

A *real hypersurface* in \mathbb{C}^{n+1} is a real submanifold of \mathbb{C}^{n+1} of real codimension one. More precisely, we say $M \subseteq \mathbb{C}^{n+1}$ is a real hypersurface if for each $p \in M$, there exists an open neighborhood $U \subseteq \mathbb{C}^{n+1}$ containing p and a smooth function $f : U \rightarrow \mathbb{R}$ such that $df \neq 0$ at any point in U and $M \cap U = \{q \in U : f(q) = 0\}$. We call such an f a *local defining function* for M near p .

Let $T_p M$ denote the real $2n + 1$ dimensional tangent space to the real hypersurface M in \mathbb{C}^{n+1} at the point $p \in M$. By identifying this with a subspace of $T_p \mathbb{R}^{2n+2}$, we may regard the complexified tangent space of M as a subspace of the complexified tangent space of \mathbb{C}^{n+1} , i.e. $\mathbb{C}T_p M \subseteq \mathbb{C}T_p \mathbb{C}^{n+1}$. We define the *CR tangent space* V_p to M at $p \in M$ to be $V_p = \mathbb{C}T_p M \cap T_p^{0,1} \mathbb{C}^{n+1}$, that is, all linear combinations of the $\frac{\partial}{\partial \bar{z}_k}$ that are also tangent to M . For a real submanifold of higher codimension, the dimension of the CR tangent space may vary from point to point, and such a submanifold is called a *CR submanifold* if the dimension does not in fact vary (equivalently, if the union of the CR tangent spaces forms a subbundle of the complexified tangent bundle of M). However, if M is a CR hypersurface and f a local defining function for M near p , by considering the restriction of df to $T_p^{0,1} \mathbb{C}^{n+1}$, we see by elementary linear algebra that the dimension of V_p will be always be n .

A smooth section of the bundle V over M is called a smooth *CR vector field*, and a smooth function on M is called a *CR function* if $Xf = 0$ for every smooth CR vector field X . In this dissertation we will assume that all objects of study (functions, vector fields, submanifolds) are smooth or real analytic, and hence will no longer specify smoothness. However, one may also study CR functions, vector fields, and submanifolds of lower regularity. We note that if h is a holomorphic function defined on an open neighborhood of M , then $h|_M$ will be CR. However, except in certain circumstances, the converse is not true.

It will be useful to have an alternate characterization of CR mappings between CR hypersurfaces. Given $M \subseteq \mathbb{C}^n$ and $M' \subseteq \mathbb{C}^N$ two CR hypersurfaces with CR bundles V and V' respectively, we say that $H : M \rightarrow M'$ is a CR mapping if $H_*(V) \subseteq V'$. It can be shown that H is a CR mapping if and only if each component of H is a CR function.

2.2 Levi Form and Hyperquadrics

We now introduce a fundamental object in the study of the geometric aspects of CR hypersurfaces, the *Levi form*. The Levi form of a CR hypersurface $M \subseteq \mathbb{C}^{n+1}$ is a Hermitian quadratic form defined at each point on the CR tangent space V to M . The Levi form allows us to introduce a notion of convexity for hypersurfaces known as *pseudoconvexity* which is fundamental in the theory of several complex variables. One well known instance of this is the *Levi problem*, which characterizes so-called domains of holomorphy in \mathbb{C}^n in terms of the pseudoconvexity of the boundary of the domain (see [Hör90] for an analytic solution or [FG02] for a more geometric solution). We also mention that the Levi form may be also seen as measuring the failure of $V \oplus \bar{V}$ to be integrable (that is, the failure of M to be foliated by complex hypersurfaces). The reader is encouraged to consult the survey articles [Tre00] and [DT10] for more information on the Levi form and pseudoconvexity, and their relationship to complex analysis in several variables.

We now give a precise definition of the Levi form. At each point $p \in M$, we define the *Levi Map*

$$\mathcal{L}_p : V_p \oplus \bar{V}_p \rightarrow \mathbb{C}T_pM / (V_p \oplus \bar{V}_p) \quad \text{by} \quad \mathcal{L}_p(X_p, Y_p) = \frac{1}{2i} \pi_p([X, \bar{Y}](p))$$

where π_p is projection from $\mathbb{C}T_pM \rightarrow \mathbb{C}T_pM / (V_p \oplus \bar{V}_p)$ and X, Y are CR vector fields on M extending X_p, Y_p . One may verify that the definition of the Levi map does not depend on the choice of CR vector fields extending X_p and Y_p . The Levi map gives a smooth bundle map from $V \oplus \bar{V} \rightarrow \mathbb{C}TM / (V \oplus \bar{V})$.

Since the complex dimension at each point of $\mathbb{C}TM$ is $2n + 1$ and that of $(V \oplus \bar{V})$ is $2n$, the quotient is one dimensional. Hence to obtain a complex valued Hermitian form from the Levi map, we need an isomorphism between the quotient and \mathbb{C} . We do this by choosing a real nonvanishing section of $(\mathbb{C}TM / (V \oplus \bar{V}))^*$, that is, a real nonvanishing one form θ on $\mathbb{C}TM$ annihilating the bundle $(V \oplus \bar{V})$. Such a form is called a *contact form* on M . Note that for an embedded M with local defining function ρ , we may choose such a θ by defining $\theta = \iota^*(i\partial\rho)$, where ι is the inclusion of M into \mathbb{C}^{n+1} and $\partial\rho$ is the projection of $d\rho$ onto $(T^{1,0}\mathbb{C}^{n+1})^*$. Therefore, given a contact form θ , a representation of the Levi map at p as a complex-valued Hermitian form is

given by $(X_p, Y_p) \rightarrow \frac{1}{2i} \langle \theta(p), [X, \bar{Y}] \rangle$, for CR vector fields X and Y extending X_p, Y_p . We mention that the Levi form is both a biholomorphic invariant (preserved by local biholomorphic mappings of the ambient complex space) and a CR invariant (preserved under CR diffeomorphisms with CR inverse).

It can be shown that given a local defining function ρ for M near p , the Levi form at p may be represented as the $(n+1) \times (n+1)$ matrix whose (j, k) entry is given by $\frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(p, \bar{p})$, restricted to the CR tangent space V_p of M at p . This suggests the relationship between the Levi form and a notion of convexity. We recall that the convexity of a real function of several variables is determined by the definiteness of the Hessian matrix of mixed second partial derivatives. In our situation the matrix of the Levi form plays a role analogous to that of the Hessian matrix of a real function, but it is restricted to the CR tangent space.

In order to discuss the notions of pseudoconvexity and strict pseudoconvexity, we must notice that there is some ambiguity in the definition of the Levi form. First, if we choose a different set of coordinates for \mathbb{C}^{n+1} , the matrix of the Levi form will be replaced by another Hermitian matrix conjugate to the original. This may change the eigenvalues of the Levi form, but by Sylvester's theorem, the signs of the eigenvalues (and number of zero eigenvalues) will be unchanged. Second, we may replace any contact form θ by $a\theta$, where a is a nonzero real number, and this will scale all eigenvalues by a . For instance, replacing θ by $-\theta$ will switch signs of all the eigenvalues. Therefore we say M is *pseudoconvex* if all nonzero eigenvalues have the same sign, and M is *strictly pseudoconvex* if all eigenvalues are nonzero and have the same sign. We say M is *Levi-nondegenerate* if the Levi form is nondegenerate (i.e. the matrix is invertible), so in particular any strictly pseudoconvex hypersurface is Levi-nondegenerate.

Let $M \subseteq \mathbb{C}^{n+1}$ be a smooth connected (as will be assumed throughout this dissertation) Levi-nondegenerate hypersurface, and let $l \leq n/2$ denote the minimum of the number of positive and negative eigenvalues of a representative of the Levi form of M at any point. This integer is constant over M and will be referred to as the *signature* of M .

We now define the standard *CR hyperquadric*, a special class of CR hypersurfaces that will figure prominently in our main results. We may think of the hyperquadric as the simplest Levi-nondegenerate hypersurface of a given signature. The reader may

consult section 1 of [CM74] for further details and properties of hyperquadrics.

Given an invertible Hermitian $n \times n$ matrix $g_{\alpha\bar{\beta}}$, where $1 \leq \alpha, \beta \leq n$, and denoting the coordinates in \mathbb{C}^{n+1} by $z^\alpha, \bar{z}^{\beta}$ ($= w = u + iv$), the hyperquadric $Q \subseteq \mathbb{C}^{n+1}$ is given by

$$v = g_{\alpha\bar{\beta}} z^\alpha \bar{z}^\beta. \quad (2.1)$$

If the matrix $g_{\alpha\bar{\beta}}$ has p negative eigenvalues and q positive eigenvalues, we say that the *signature pair* of Q is (p, q) . Notice that at the origin, the matrix $g_{\alpha\bar{\beta}}$ is the matrix of the Levi form of Q , hence Q is Levi-nondegenerate and connected, with signature $\min(p, q)$ and CR dimension n . The linear fractional transformation

$$Z^\alpha = \frac{2z^\alpha}{w+i}, \quad W = \frac{w-i}{w+i}$$

takes (2.1) to

$$g_{\alpha\bar{\beta}} Z^\alpha \bar{Z}^\beta + |W|^2 = 1. \quad (2.2)$$

By a complex linear change of coordinates we may replace $g_{\alpha\bar{\beta}}$ by the matrix with p negative ones and q positive ones on the diagonal, and zero entries everywhere else. It is then clear that if $g_{\alpha\bar{\beta}}$ is positive definite, equation (2.2) describes a hypersphere in \mathbb{C}^{n+1} .

We will generally embed our CR hyperquadrics in complex projective space, as this will be important when we discuss Q -frames (see Chapter 3). We embed \mathbb{C}^{n+1} in $\mathbb{C}\mathbb{P}^{n+1}$ as the set $\zeta^0 \neq 0$ in homogeneous coordinates $[\zeta^0, \dots, \zeta^{n+1}]$ on $\mathbb{C}\mathbb{P}^{n+1}$. In these homogeneous coordinates, with $Z^i = \frac{\zeta^i}{\zeta^0}$ for $1 \leq i \leq n+1$, after replacing $g_{\alpha\bar{\beta}}$ with the aforementioned diagonal matrix and letting $p = l$, (2.2) becomes

$$-\sum_{j=0}^l |\zeta_j|^2 + \sum_{k=l+1}^{n+1} |\zeta_k|^2 = 0. \quad (2.3)$$

This is the closure of the quadric given by (2.2) in projective space (i.e. the quadric given by (2.2) together with the ‘‘quadric at infinity’’). Equation (2.3) will be the standard representation of the quadric that we utilize. We denote this quadric by Q_l^n , so the superscript is the CR dimension and the subscript is the signature (we may multiply the equation by negative one if necessary to ensure $l \leq n/2$). When $l = 0$, the quadric is equivalent to the standard sphere S^{2n+1} embedded in projective space.

2.3 Embeddings, Coframes, and the Tanaka-Webster Connection

Of fundamental importance in the study of embeddings of CR hypersurfaces are appropriate frames and coframes for the complexified tangent space to a hypersurface M and the complexified dual space, as well as a suitable connection. We introduce these objects and some of their properties in this section. In what follows, we will work with a fixed choice of contact form. The study of geometric properties of a hypersurface with fixed choice of contact form is known as *pseudo-Hermitian* geometry, and the study of the general CR properties of a hypersurface invariant under local biholomorphic mappings (i.e. without a fixed choice of contact form) is known as *pseudo-conformal* geometry. Recall that our considerations are all local, so we are assumed to be working in a sufficiently small neighborhood of a point.

Let M be a Levi-nondegenerate CR hypersurface of dimension $2n + 1$, with rank n CR bundle V and signature $l \leq n/2$. Near a point p_0 , we let θ be a contact form and T its *characteristic* (or Reeb) vector field, so T is the unique real vector field satisfying $T \lrcorner d\theta = 0$ and $\langle \theta, T \rangle = 1$. We complete θ to an admissible coframe $(\theta, \theta^1, \dots, \theta^n)$ for the bundle $T'M$ of $(1, 0)$ -cotangent vectors (the cotangent vectors that annihilate V). The coframe is called *admissible* if $\langle \theta^\alpha, T \rangle = 0$, for $\alpha = 1, \dots, n$.

We choose a frame L_1, \dots, L_n for the bundle \bar{V} such that $(T, L_1, \dots, L_n, L_{\bar{1}}, \dots, L_{\bar{n}})$ is a frame for $\mathbb{C}TM$ dual to the coframe $(\theta, \theta^1, \dots, \theta^n, \theta^{\bar{1}}, \dots, \theta^{\bar{n}})$. We use the notation that $L_{\bar{\alpha}} = \bar{L}_\alpha$, etc. Relative to this frame, let $(g_{\alpha\bar{\beta}})$ denote the matrix of the Levi form. Although we generally won't explicitly use this fact, we may assume $g_{\alpha\bar{\beta}}$ is constant and diagonal, with diagonal elements ± 1 corresponding to the signature.

Recall that a connection on a vector bundle is a structure that allows one to differentiate sections of the bundle with respect to other sections. Generally such a connection is chosen to be compatible with some specified metric. Specifically, given a vector bundle $B \rightarrow M$, and letting $\Gamma(B)$ denote the space of smooth sections of the bundle, a connection is a linear map $\nabla : \Gamma(B) \rightarrow \Gamma(B \otimes T^*M)$ satisfying the Leibniz rule $\nabla(\sigma f) = f\nabla\sigma + \sigma \otimes df$ for smooth functions f on M and smooth sections σ of the bundle B . In Riemannian geometry, the Levi-Civita connection on an abstract Riemannian

manifold is the unique torsion free connection on the tangent bundle that is compatible with the metric. The Tanaka-Webster connection will satisfy similar properties.

We denote the Tanaka-Webster connection by ∇ , given relative to the chosen frame and coframe by

$$\nabla L_\alpha := \omega_\alpha^\beta \otimes L_\beta.$$

The connection 1-forms ω_α^β are completely determined by the conditions

$$\begin{aligned} d\theta^\beta &= \theta^\alpha \wedge \omega_\alpha^\beta \quad \text{mod } \theta \wedge \theta^{\bar{\alpha}}, \\ dg_{\alpha\bar{\beta}} &= \omega_{\alpha\bar{\beta}} + \omega_{\bar{\beta}\alpha}. \end{aligned} \quad (2.4)$$

Note that we use the Levi form to lower and raise indices as usual. Although the connection was originally defined only for sections of $T^{1,0}M$, we may define it on \mathcal{CTM} by requiring $\nabla T = 0$ and $\nabla L_{\bar{\alpha}} := \omega_{\bar{\alpha}}^\beta \otimes L_{\bar{\beta}}$, where $\omega_{\bar{\alpha}}^\beta = \overline{\omega_\alpha^\beta}$.

We may rewrite the first condition in (2.4) as

$$d\theta^\beta = \theta^\alpha \wedge \omega_\alpha^\beta + \theta \wedge \tau^\beta, \quad \tau^\beta = A_{\bar{\nu}}^\beta \theta^{\bar{\nu}}, \quad A^{\alpha\beta} = A^{\beta\alpha} \quad (2.5)$$

for a suitably determined torsion matrix $(A_{\bar{\nu}}^\beta)$, where the last symmetry relation holds automatically (see [W78]). We also notice that the coframe $(\theta, \theta^1, \dots, \theta^n)$ is admissible if and only if $d\theta = ig_{\alpha\bar{\beta}} \theta^\alpha \wedge \theta^{\bar{\beta}}$, and it is this characterization of admissibility that we will use.

Now let \hat{M} be another Levi-nondegenerate CR hypersurface of dimension $2\hat{n} + 1$, with rank \hat{n} CR bundle \hat{V} and signature $\hat{l} \leq \hat{n}/2$. Let $f: M \rightarrow \hat{M}$ be a smooth CR mapping in a small neighborhood of p_0 . Since our arguments are local in nature, we denote this neighborhood by M also. We use a $\hat{\cdot}$ to denote objects associated to \hat{M} . Capital Latin indices A, B , etc. will belong to the set $\{1, \dots, \hat{n}\}$, Greek indices α, β , etc. will belong to $\{1, \dots, n\}$, and small Latin indices a, b , etc. run over the complementary set $\{n+1, \dots, \hat{n}\}$. Let (θ, θ^α) and $(\hat{\theta}, \hat{\theta}^A)$ be coframes on M and \hat{M} respectively. Note that f is a CR mapping if and only if

$$f^*(\hat{\theta}) = a\theta, \quad f^*(\hat{\theta}^A) = E_\alpha^A \theta^\alpha + E^A \theta, \quad (2.6)$$

where a is a real-valued function and E_α^A, E^A are complex-valued functions defined near p_0 . This is because in coordinates, the pushforward matrix of f is a matrix with real entries, so if $v \in T^{1,0}M$, then $f_*(v) = \overline{f_*(\bar{v})} \in T^{1,0}\hat{M}$.

An important notion that we will need is that of CR transversality. Recall that given a smooth real manifold $N \subseteq \mathbb{R}^k$ and $p \in N$, a smooth map $f : \mathbb{R}^l \rightarrow \mathbb{R}^k$ is said to be *transversal* to N at p if $f_*(T_{f^{-1}(p)}\mathbb{R}^l) + T_p N = T_p \mathbb{R}^k$. Now given $p \in \hat{M}$ where \hat{M} is a CR hypersurface in $\mathbb{C}^{\hat{n}+1}$, we say that a holomorphic mapping $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{\hat{n}+1}$ is *CR transversal* to \hat{M} at p if $f_*(T_{f^{-1}(p)}^{0,1}\mathbb{C}^{n+1}) + T_p^{0,1}\hat{M} = T_p^{0,1}\mathbb{C}^{\hat{n}+1}$. Note that if f is CR transversal to \hat{M} at p then considered as a smooth map from $\mathbb{R}^{2n+2} \rightarrow \mathbb{R}^{2\hat{n}+2}$, f is transversal to \hat{M} at p . In general the converse is not true, and we refer the reader to the papers [ER06] and [DE10] for further details on CR transversality. As a consequence of the following lemma, if f is as in equation (2.6) above and $f^*(\hat{\theta}) = a\theta$, then $a \neq 0$.

Lemma 2.3.1. *If $M \subseteq \mathbb{C}^{n+1}$ and $\hat{M} \subseteq \mathbb{C}^{\hat{n}+1}$ are CR hypersurfaces, $p \in \hat{M}$, and $f : \mathbb{C}^{n+1} \rightarrow \mathbb{C}^{\hat{n}+1}$ is a germ of a holomorphic CR transversal mapping at p with $f(M) \subseteq \hat{M}$, then if $f^*(\hat{\theta}) = a\theta$, we have $a(p) \neq 0$.*

Proof. Since we know that

$$f_*(T_{f^{-1}(p)}^{0,1}\mathbb{C}^{n+1}) + T_p^{0,1}\hat{M} = T_p^{0,1}\mathbb{C}^{\hat{n}+1},$$

conjugation yields

$$f_*(T_{f^{-1}(p)}^{1,0}\mathbb{C}^{n+1}) + T_p^{1,0}\hat{M} = T_p^{1,0}\mathbb{C}^{\hat{n}+1},$$

and hence

$$f_*(\mathbb{C}T_{f^{-1}(p)}\mathbb{C}^{n+1}) + T_p^{0,1}\hat{M} \oplus T_p^{1,0}\hat{M} = \mathbb{C}T_p\mathbb{C}^{\hat{n}+1}.$$

Let L denote the span of the one dimensional transverse space to $\mathbb{C}T_{f^{-1}(p)}M$ in $\mathbb{C}T_{f^{-1}(p)}\mathbb{C}^{n+1}$, T the complex span of the Reeb vector field for M at $f^{-1}(p)$, and V and \bar{V} the CR tangent space to M at $f^{-1}(p)$ and its conjugate. Then we have

$$f_*(T_{f^{-1}(p)}^{0,1}M \oplus T_{f^{-1}(p)}^{1,0}M) + f_*(T) + f_*(L) + T_p^{0,1}\hat{M} \oplus T_p^{1,0}\hat{M} = \mathbb{C}T_p\mathbb{C}^{\hat{n}+1}.$$

Since f is CR, we must have that $f_*(T_{f^{-1}(p)}^{0,1}M \oplus T_{f^{-1}(p)}^{1,0}M) \subseteq T_p^{0,1}\hat{M} \oplus T_p^{1,0}\hat{M}$, so if $a(p) = 0$, then $f^*(\hat{\theta}) = 0$, so $f_*(T) \subseteq T_p^{0,1}\hat{M} \oplus T_p^{1,0}\hat{M}$. Therefore

$$f_*(L) + T_p^{0,1}\hat{M} \oplus T_p^{1,0}\hat{M} = \mathbb{C}T_p\mathbb{C}^{\hat{n}+1},$$

but this cannot occur because the left side is one complex dimension lower than the right side. \square

Given an admissible coframe $(\hat{\theta}, \hat{\theta}^A)$ for \hat{M} , we may apply f^* to the equation $d\hat{\theta} = i g_{AB} \hat{\theta}^A \wedge \hat{\theta}^B$. The CR transversality of f implies that $g_{\alpha\beta} = \frac{1}{a} \hat{g}_{AB} E^A_\alpha E^B_\beta$. Therefore $n \leq \hat{n}$ and f is locally an embedding.

Now suppose (θ, θ^α) is a coframe on M such that the matrix of the Levi form with respect to this coframe has l negative and $n - l$ positive eigenvalues. Let $(\hat{\theta}, \hat{\theta}^A)$ be a coframe on \hat{M} such that the matrix of the Levi form with respect to this coframe has l' negative and $\hat{n} - l'$ positive eigenvalues. If $l < n/2$ and $l' < \hat{n}/2$, we define f to be *side preserving* if the nonvanishing function a such that $f^*(\hat{\theta}) = a\theta$ is strictly positive, and *side reversing* if a is strictly negative. Note that this definition does not depend on the choice of pseudohermitian structure.

We state the following result, which is essentially Proposition 3.1 in [BEH08] although we have been careful to distinguish the side preserving and side reversing cases.

Proposition 2.3.1. *Let M and \hat{M} be Levi-nondegenerate CR-manifolds of dimensions $2n + 1$ and $2\hat{n} + 1$, and signatures $l \leq n/2$ and $l' \leq \hat{n}/2$ respectively. Let $f : M \rightarrow \hat{M}$ be a CR mapping that is CR transversal to \hat{M} along M . If (θ, θ^α) is any admissible coframe on M , then in a neighborhood of any point $\hat{p} \in f(M)$ in \hat{M} there exists an admissible coframe $(\hat{\theta}, \hat{\theta}^A)$ on \hat{M} with $f^*(\hat{\theta}, \hat{\theta}^\alpha, \hat{\theta}^A) = (\theta, \theta^\alpha, 0)$. If the Levi form of M with respect to (θ, θ^α) is constant and diagonal with $-1, \dots, -1$ (l times) and $1, \dots, 1$ ($n - l$ times) on the diagonal, then $(\hat{\theta}, \hat{\theta}^A)$ can be chosen such that the Levi form of \hat{M} relative to this coframe is constant and diagonal and if f is*

side preserving or $l = n/2$ or $l' = \hat{n}/2$, the diagonal elements are $-1, \dots, -1$ (l times), $1, \dots, 1$ ($n - l$ times), $-1, \dots, -1$ ($l' - l$ times) and $1, \dots, 1$ ($\hat{n} - n - l' + l$ times). With this additional property, the coframe $(\hat{\theta}, \hat{\theta}^A)$ is uniquely determined along M up to unitary transformations in $U(n, l) \times U(\hat{n} - n, l' - l)$.

side reversing, the diagonal elements are $-1, \dots, -1$ (l times), $1, \dots, 1$ ($n - l$ times), $-1, \dots, -1$ ($\hat{n} - l' - l$ times) and $1, \dots, 1$ ($l' - (n - l)$ times). With this additional property, the coframe $(\hat{\theta}, \hat{\theta}^A)$ is uniquely determined along M up to unitary transformations in $U(n, l) \times U(\hat{n} - n, \hat{n} - l' - l)$.

Observe that if $l = n/2$ we may change the sign of θ so that the Levi form

resembles the side preserving case. If $l' = \hat{n}/2$, the two conclusions of the proposition coincide. If we fix an admissible coframe (θ, θ^α) on M and let $(\hat{\theta}, \hat{\theta}^A)$ be an admissible coframe on \hat{M} near a point $\hat{p} \in f(M)$, we shall say $(\hat{\theta}, \hat{\theta}^A)$ is *adapted* to (θ, θ^α) on M (or just to M if the coframe on M is understood) if it satisfies the conclusions of Proposition 3.1 with the requirement there for the Levi form. However we will continue to write the Levi forms as $g_{\alpha\bar{\beta}}, \hat{g}_{A\bar{B}}$. We shall also omit the $\hat{\cdot}$ over frames and coframes if there is no ambiguity. It will be clear from the context if a form is pulled back to M or not. Under the above assumptions, we identify M with the submanifold $f(M)$ and write $M \subset \hat{M}$.

2.4 The Second Fundamental Form, Covariant Derivatives, and Degenerate Maps

Equation (2.5) implies that when (θ, θ^A) is adapted to M , if the pseudoconformal connection matrix of $(\hat{M}, \hat{\theta})$ is $\hat{\omega}_B^A$, then that of (M, θ) is the pullback of $\hat{\omega}_\beta^\alpha$. The pulled back torsion $\hat{\tau}^\alpha$ is τ^α , so omitting the $\hat{\cdot}$ over these pullbacks will not cause any ambiguity and we shall do that from now on. By the normalization of the Levi form, the second equation in (2.4) reduces to

$$\omega_{B\bar{A}} + \omega_{\bar{A}B} = 0, \quad (2.7)$$

where as before $\omega_{\bar{A}B} = \overline{\omega_{AB}}$.

An important concept from Riemannian geometry that we will use in the CR setting is the *second fundamental form*. Recall that if M is a Riemannian hypersurface isometrically embedded in a Riemannian manifold \hat{M} , then the second fundamental form is given by the projection of the ambient connection onto the normal bundle of M in \hat{M} . We may therefore interpret the second fundamental form as measuring the difference between the connection on M and the connection on \hat{M} restricted to M . There is also a notion of a CR Gauss equation, which we will introduce and make precise in chapter 5.

We define the second fundamental form of the embedding $f : M \rightarrow \hat{M}$ as the matrix of 1-forms (ω_α^b) pulled back to M . Since $\theta^b = 0$ on M , equation (2.5) implies that on M ,

$$\omega_\alpha^b \wedge \theta^\alpha + \tau^b \wedge \theta = 0, \quad (2.8)$$

and this implies that

$$\omega_\alpha^b = \omega_{\alpha\beta}^b \theta^\beta, \quad \omega_{\alpha\beta}^b = \omega_\beta^b{}_\alpha, \quad \tau^b = 0. \quad (2.9)$$

We refer the reader to [EHZ04] for a further discussion of the extrinsic and intrinsic notion of second fundamental form.

Again following [EHZ04] we identify the CR-normal space $T_p^{1,0}\hat{M}/T_p^{1,0}M$, also denoted by $N_p^{1,0}\hat{M}$ with $\mathbb{C}^{\hat{n}-n}$ by choosing the equivalence classes of L_a as a basis. Therefore for fixed $\alpha, \beta = 1, \dots, n$, we view the component vector $(\omega_{\alpha\beta}^a)_{a=n+1, \dots, \hat{n}}$ as an element of $\mathbb{C}^{\hat{n}-n}$. By also viewing the second fundamental form as a section over M of the bundle $T^{1,0}M \otimes N^{1,0}\hat{M} \otimes T^{1,0}M$, we may use the pseudohermitian connections on M and \hat{M} to define the covariant differential

$$\nabla \omega_{\alpha\beta}^a = d\omega_{\alpha\beta}^a - \omega_\mu^a{}_\beta \omega_\alpha^\mu + \omega_{\alpha\beta}^b \omega_b^a - \omega_{\alpha\mu}^a \omega_\beta^\mu.$$

We write $\omega_{\alpha\beta;\gamma}^a$ to denote the component in the direction θ^γ and define higher order derivatives inductively as:

$$\nabla \omega_{\gamma_1 \gamma_2 \gamma_3 \dots \gamma_j}^a = d\omega_{\gamma_1 \gamma_2 \gamma_3 \dots \gamma_j}^a + \omega_{\gamma_1}^b{}_{\gamma_2 \gamma_3 \dots \gamma_j} \omega_b^a - \sum_{l=1}^j \omega_{\gamma_1 \gamma_2 \gamma_3 \dots \gamma_{l-1} \mu \gamma_{l+1} \dots \gamma_j} \omega_{\gamma_l}^\mu. \quad (2.10)$$

We also consider the component vectors of higher order derivatives as elements of $\mathbb{C}^{\hat{n}-n}$ and define an increasing sequence of vector spaces

$$E_2(p) \subseteq \dots \subseteq E_k(p) \subseteq \dots \subseteq \mathbb{C}^{\hat{n}-n}$$

by letting $E_k(p)$ be the span of the vectors

$$(\omega_{\gamma_1 \gamma_2 \gamma_3 \dots \gamma_j}^a)_{a=n+1, \dots, \hat{n}}, \quad \forall 2 \leq j \leq k, \gamma_j \in \{1, \dots, n\},$$

evaluated at $p \in M$. Following Lamel [La01] and [EHZ04], we say that the mapping $f : M \rightarrow \hat{M}$ is *constantly (k, s) -degenerate* at p if the vector space $E_k(p)$ has constant dimension $\hat{n} - n - s$ for q near p , $E_{k+1}(q) = E_k(q)$, and k is the smallest such integer.

2.5 Two Important Lemmas

We now state two key lemmas that are ingredients in the proofs of subsequent theorems. The first lemma appears in [Hu99] as Lemma 2.1 with $H = I$. Although the

proof for arbitrary constant invertible H is identical, we reproduce it here for the reader's convenience.

Lemma 2.5.1. *Let $g_1, \dots, g_k, f_1, \dots, f_k$ be holomorphic functions in $z \in \mathbb{C}^n$ near 0. Assume $g_j(0) = f_j(0) = 0$ for all j . Let $A(z, \bar{z})$ be real-analytic near the origin such that*

$$\sum_{j=1}^k g_j(z) \overline{f_j(z)} = A(z, \bar{z}) (h_{a\bar{b}} z^a \bar{z}^{\bar{b}})$$

where $H = (h_{a\bar{b}})$ is a constant invertible matrix. If $k < n$, then $A(z, \bar{z}) \equiv 0$.

Proof. Complexify the above identity to obtain $\sum_{j=1}^k \psi_j(z) \overline{\chi_j(\bar{\zeta})} = \langle z, \bar{\zeta} \rangle_g H(z, \zeta)$. Without loss of generality we may assume no ψ_j is identically zero, so there exists a point z_0 close to the origin such that $\psi_j(z_0) = \varepsilon_j \neq 0$ for all j .

Define $V_{z_0} = \{z : \psi_j(z) = \varepsilon_j, j = 1, \dots, k\}$, a complex analytic variety of dimension greater than or equal to one, by the assumption $k < n$. Since all ψ_j vanish at the origin, $0 \notin V_{z_0}$, so V_{z_0} cannot contain a complex line through the origin or be contained in such a line. This is true because if V_{z_0} contains a piece of a complex line L through the origin, it must contain the whole line by elementary facts about analytic varieties.

Hence there exists a $z^* \in V_{z_0}$ such that V_{z_0} contains a complex curve C^* near z^* parametrized by $z(t) = z^* + vt + o(t)$, where z^* and v are independent and $|t| < 1$.

Notice that for each $z \in C^*$ and ζ such that $\langle z, \bar{\zeta} \rangle_g = 0$ with ζ close to the origin, we have $\sum_{j=1}^k \varepsilon_j \overline{\chi_j(\bar{\zeta})} = 0$.

Now $\langle z, \bar{\zeta} \rangle_g = \zeta^T Gz$, which is zero when $\bar{\zeta}$ is orthogonal to Gz . Since G is an invertible linear map, $G(C^*)$ is a complex curve that isn't contained in a line through the origin, and doesn't contain the origin. Hence the set of all ζ such that $\langle z, \bar{\zeta} \rangle_g = 0$ and $z \in C^*$ contains an open set, so $\sum_{j=1}^k \varepsilon_j \overline{\chi_j(\bar{\zeta})} \equiv 0$.

We then solve for χ_k and reduce the initial equation to

$$\sum_{j=1}^{k-1} \left(\psi_j(z) - \frac{\varepsilon_j}{\varepsilon_k} \psi_k(z) \right) \overline{\chi_j(z)} = \langle z, z \rangle_g H(z, \bar{z}).$$

The proof now follows by induction. □

Lemma 2.5.2. *Let k, l, n be nonnegative integers with $k < l \leq n/2$. Assume that $g_1, \dots, g_k, f_1 \dots f_m$ are germs at $0 \in \mathbb{C}^n$ of holomorphic functions and $A(z, \bar{z})$ be real-analytic near the origin*

such that

$$-\sum_{i=1}^k |g_i(z)|^2 + \sum_{j=1}^m |f_j(z)|^2 = A(z, \bar{z}) \left(-\sum_{i=1}^l |z_i|^2 + \sum_{j=l+1}^n |z_j|^2 \right).$$

Then $A(z, \bar{z}) \equiv 0$.

The proof of Lemma 2.2 can be found in Lemma 4.1 of [BH05] (with $l' = l$ and after an application of Lemma 2.1 of [BH05]).

Chapter 3

Chern-Moser and Webster Theory

In this chapter we introduce additional concepts from Chern and Moser's theory of pseudoconformal invariants and Webster's theory of pseudo-Hermitian invariants. We refer the reader to the celebrated papers [CM74] and [W78] for details of proofs and derivations.

A fundamental problem in the local geometric theory of several complex variables is to determine if two real hypersurfaces M and M' in \mathbb{C}^{n+1} are *pseudoconformally equivalent*. This means that there exists a biholomorphic map from an open neighborhood of M to an open neighborhood of M' whose restriction to M is a homeomorphism between M and M' . We notice that if M and M' are pseudoconformally equivalent then they are *CR equivalent*, meaning that there exists a CR map from M to M' which is a diffeomorphism.

This problem was solved in \mathbb{C}^2 by E. Cartan and then in higher dimensions by S.S. Chern and J. Moser. We remark that Tanaka (see [T76]) independently obtained a solution which was not published until after Chern and Moser's solution was published. The idea of Chern and Moser's proof is, given a Levi-nondegenerate hypersurface M with signature (p, q) , to construct a principal fiber bundle Y over M whose structure group H is the subgroup of $SU(p+1, q+1)$ fixing the point $(1, 0, \dots, 0) \in \mathbb{C}^{n+2}$. There is also a Cartan connection ω with values in the Lie algebra of $SU(p+1, q+1)$. Then M and M' are pseudoconformally equivalent if and only if there exists a bundle map $\phi : Y \rightarrow Y'$ such that $\phi^*(\omega') = \omega$, where Y' and ω' are the corresponding bundle and Cartan connection for M' . We may think of ω as giving a list of pseudoconformal

invariants for M .

S. Webster then considered so-called *pseudo-Hermitian* manifolds. Such a manifold consists of a pair (M, θ) , where M is a real CR hypersurface and θ is a fixed choice of contact form for M . Webster performed a construction similar to that of Chern and Moser to obtain a list of pseudo-Hermitian invariants to solve the equivalence problem for pseudo-Hermitian manifolds. Although Webster's pseudo-Hermitian invariants are generally not CR invariants (that is, preserved by CR diffeomorphisms), there is a relationship between the pseudo-Hermitian and pseudoconformal invariants which we will make extensive use of.

In the first section of this chapter we list structure equations for Chern and Moser's pseudoconformal invariants, and introduce their notion of pseudoconformal curvature. We then present the relevant relationships between the Webster and Chern-Moser invariants, as well as the concept of a conformally flat tensor. Finally, we introduce Q-frames, which will play a key role in the proof of Theorem 1.

3.1 The Structure Equations and Pseudoconformal Curvature

We present the pseudoconformal connection and structure equations introduced in [CM74]. Let Y be the bundle of coframes $(\omega, \omega^\alpha, \omega^{\bar{\alpha}}, \phi)$ on the real ray bundle $\pi_E : E \rightarrow M$ of all contact forms defining the same orientation of M , such that $d\omega = ig_{\alpha\bar{\beta}}\omega^\alpha \wedge \omega^{\bar{\beta}} + \omega \wedge \phi$ where $\omega^\alpha \in \pi_E^*(T^*M)$ and ω is the canonical 1-form on E . In [CM74] it was shown that these forms can be completed to a full set of invariants on Y given by the coframe of 1-forms

$$(\omega, \omega^\alpha, \omega^{\bar{\alpha}}, \phi, \phi_\beta^\alpha, \phi^\alpha, \phi^{\bar{\alpha}}, \psi)$$

which define the pseudoconformal connection on Y . These forms satisfy the following structure equations (see [CM74] and its appendix):

$$\begin{aligned}
\phi_{\alpha\bar{\beta}} + \phi_{\bar{\beta}\alpha} &= g_{\alpha\bar{\beta}}\phi, \\
d\omega &= i\omega^\mu \wedge \omega_\mu + \omega \wedge \phi, \\
d\omega^\alpha &= \omega^\mu \wedge \phi_\mu^\alpha + \omega \wedge \phi^\alpha, \\
d\phi &= i\omega_{\bar{\nu}} \wedge \phi^{\bar{\nu}} + i\phi_{\bar{\nu}} \wedge \omega^{\bar{\nu}} + \omega \wedge \psi, \\
d\phi_\beta^\alpha &= \phi_\beta^\mu \wedge \phi_\mu^\alpha + i\omega_\beta \wedge \phi^\alpha - i\phi_\beta \wedge \omega^\alpha - i\delta_\beta^\alpha \phi_\mu \wedge \omega^\mu - \frac{\delta_\beta^\alpha}{2}\psi \wedge \omega + \Phi_\beta^\alpha, \\
d\phi^\alpha &= \phi \wedge \phi^\alpha + \phi^\mu \wedge \phi_\mu^\alpha - \frac{1}{2}\psi \wedge \omega^\alpha + \Phi^\alpha, \\
d\psi &= \phi \wedge \psi + 2i\phi^\mu \wedge \phi_\mu + \Psi.
\end{aligned} \tag{3.1}$$

Here the 2-forms $\Phi_\beta^\alpha, \Phi^\alpha, \Psi$ give the *pseudoconformal curvature* of M . We may decompose Φ_β^α as follows

$$\Phi_\beta^\alpha = S_{\beta\ \mu\bar{\nu}}^\alpha \omega^\mu \wedge \omega^{\bar{\nu}} + V_{\beta\ \mu}^\alpha \omega^\mu \wedge \omega + V_{\beta\bar{\nu}}^\alpha \omega \wedge \omega^{\bar{\nu}}.$$

We will also refer to the tensor $S_{\beta\ \mu\bar{\nu}}^\alpha$ as the pseudoconformal curvature of M . We only consider coframes (θ, θ^α) for which the Levi form is constant. We require $S_{\beta\ \mu\bar{\nu}}^\alpha$ to satisfy certain trace and symmetry conditions (see [CM74]), but for the purposes of this paper, the important point to emphasize is that *for a hyperquadric, the pseudoconformal curvature vanishes*.

3.2 Pseudo-Hermitian Invariants and Conformally Flat Tensors

If we fix a contact form θ that defines a section $M \rightarrow E$, then any admissible coframe (θ, θ^α) for M defines a unique section $M \rightarrow Y$ under which the pullbacks of (ω, ω^α) coincide with (θ, θ^α) and the pullback of ϕ vanishes. As in [W78] we use this section to pull the pseudoconformal connection forms back to M . Although the pulled back forms on M now depend on the choice of admissible coframe, we shall use the same notation, and thus we have

$$\theta = \omega, \quad \theta^\alpha = \omega^\alpha, \quad \phi = 0$$

on M . As in [W78], we may write the pulled back tangential pseudoconformal curvature tensor $S_{\beta\ \mu\bar{\nu}}^{\alpha}$ in terms of the tangential pseudo-Hermitian curvature tensor $R_{\beta\ \mu\bar{\nu}}^{\alpha}$ by

$$S_{\alpha\bar{\beta}\mu\bar{\nu}} = R_{\alpha\bar{\beta}\mu\bar{\nu}} - \frac{R_{\alpha\bar{\beta}}g_{\mu\bar{\nu}} + R_{\mu\bar{\beta}}g_{\alpha\bar{\nu}} + R_{\alpha\bar{\nu}}g_{\mu\bar{\beta}} + R_{\mu\bar{\nu}}g_{\alpha\bar{\beta}}}{n+2} + \frac{R(g_{\alpha\bar{\beta}}g_{\mu\bar{\nu}} + g_{\alpha\bar{\nu}}g_{\mu\bar{\beta}})}{(n+1)(n+2)},$$

where

$$R_{\alpha\bar{\beta}} := R_{\mu\ \alpha\bar{\beta}}^{\mu} \quad \text{and} \quad R := R_{\mu}^{\mu}$$

are respectively the pseudo-Hermitian Ricci and scalar curvature of (M, θ) . This formula expresses the fact that $S_{\alpha\bar{\beta}\mu\bar{\nu}}$ is the “traceless component” of $R_{\alpha\bar{\beta}\mu\bar{\nu}}$ with respect to the decomposition of the space of all tensors with the symmetry conditions of $S_{\alpha\bar{\beta}\mu\bar{\nu}}$ into the direct sum of the subspace of tensors with trace zero and the subspace of *conformally flat tensors*, i.e. tensors of the form

$$T_{\alpha\bar{\beta}\mu\bar{\nu}} = H_{\alpha\bar{\beta}}g_{\mu\bar{\nu}} + H_{\mu\bar{\beta}}g_{\alpha\bar{\nu}} + H_{\alpha\bar{\nu}}g_{\mu\bar{\beta}} + H_{\mu\bar{\nu}}g_{\alpha\bar{\beta}}, \quad (3.2)$$

where $(H_{\alpha\bar{\beta}})$ is any Hermitian matrix. We shall call two tensors as above *conformally equivalent* if their difference is of the form of equation (3.2). Note that covariant derivatives of conformally flat tensors are conformally flat, because $\nabla g_{\alpha\bar{\beta}} = 0$.

The following result relates the pseudoconformal and pseudo-Hermitian connection forms. It is alluded to in [W78] and a proof may be found in [EHZ04], where the result appears as Proposition 3.1. Note that although the Proposition in [EHZ04] is stated only for M strictly pseudoconvex, the result is valid in the Levi-nondegenerate situation.

Proposition 3.2.1. *Let M be a smooth Levi-nondegenerate CR-manifold of hypersurface type with CR dimension n , and with respect to an admissible coframe $(\theta, \theta^{\alpha})$ let the pseudoconformal and pseudo-Hermitian connection forms be pulled back to M as above. Then we have the following relations:*

$$\phi_{\beta}^{\alpha} = \omega_{\beta}^{\alpha} + D_{\beta}^{\alpha}\theta, \quad \phi^{\alpha} = \tau^{\alpha} + D_{\mu}^{\alpha}\theta^{\mu} + E^{\alpha}\theta, \quad \psi = iE_{\mu}\theta^{\mu} - iE_{\bar{\nu}}\theta^{\bar{\nu}} + B\theta,$$

where

$$\begin{aligned} D_{\alpha\bar{\beta}} &:= \frac{iR_{\alpha\bar{\beta}}}{n+2} - \frac{iRg_{\alpha\bar{\beta}}}{2(n+1)(n+2)}, \\ E^\alpha &:= \frac{2i}{2n+1}(A^{\alpha\mu}_{;\mu} - D^{\bar{\nu}\alpha}_{;\bar{\nu}}), \\ B &:= \frac{1}{n}(E^\mu_{;\mu} + E^{\bar{\nu}}_{;\bar{\nu}} - 2A^{\beta\mu}A_{\beta\mu} + 2D^{\bar{\nu}\alpha}D_{\bar{\nu}\alpha}). \end{aligned}$$

3.3 Q-Frames

Another notion that will prove useful is that of an adapted Q -frame. We embed \mathbb{C}^{N+1} in $\mathbb{C}\mathbb{P}^{N+1}$ as the set $\{\zeta^0 \neq 0\}$ in the homogeneous coordinates $[\zeta^0 : \zeta^1 : \dots : \zeta^{N+1}]$, and following section 1 of [CM74], realize the quadric Q_l^N given in $\mathbb{C}\mathbb{P}^{N+1}$ by the equation $(\zeta, \zeta) = 0$, where the Hermitian scalar product (\cdot, \cdot) is defined by

$$(\zeta, \tau) := \hat{I}_{A\bar{B}}\zeta^A\bar{\tau}^B + \frac{i}{2}(\zeta^{N+1}\bar{\tau}^0 - \zeta^0\bar{\tau}^{N+1}). \quad (3.3)$$

In the above, \hat{I} is the diagonal matrix with first l diagonal entries equal to -1 and all subsequent diagonal entries equal 1, and capital Latin indices run from 1 to N . Note that equation (3.3) is derived by introducing the coordinates $z^i = \frac{\zeta^i}{\zeta^0}$, $1 \leq i \leq N+1$ in equation (2.1).

A Q -frame (see e.g. [CM74]) is a unimodular basis (Z_0, \dots, Z_{N+1}) of \mathbb{C}^{N+2} (that is, $\det(Z_0, \dots, Z_{N+1}) = 1$) such that Z_0 and Z_{N+1} , as points in $\mathbb{C}\mathbb{P}^{N+1}$, are on Q , the scalar product $(Z_A, Z_B) = 0$ if $A \neq B$, $(Z_A, Z_B) = -1$ if $1 \leq A = B \leq l$, $(Z_A, Z_B) = 1$ if $l+1 \leq A = B \leq N$, and $(Z_{N+1}, Z_0) = i/2$. We will denote the corresponding points in $\mathbb{C}\mathbb{P}^{N+1}$ also by Z_0 and Z_{N+1} ; it should be clear from the context whether the point is in \mathbb{C}^{N+2} or $\mathbb{C}\mathbb{P}^{N+1}$.

On the space \mathfrak{B} of all Q -frames there is a natural action of the group $\mathbf{SU}(l+1, \hat{n}-l+1)$ of unimodular $(\hat{n}+2) \times (\hat{n}+2)$ matrices that preserve the inner product (3.3). A matrix $M \in \mathbf{SU}(l+1, \hat{n}-l+1)$ acts on a Q -frame Q_1 by left multiplication with the matrix whose columns are, in order, the elements of Q_1 . It is easy to see that this action is both free and transitive. Hence, any fixed Q -frame defines an isomorphism between \mathfrak{B} and $\mathbf{SU}(l+1, \hat{n}-l+1)$. On the space \mathfrak{B} , there are *Maurer-Cartan forms*

π_Λ^Ω , where capital Greek indices run from 0 to $N + 1$, defined by

$$dZ_\Lambda = \pi_\Lambda^\Omega Z_\Omega \quad (3.4)$$

and satisfying $d\pi_\Lambda^\Omega = \pi_\Lambda^\Gamma \wedge \pi_\Gamma^\Omega$. Here the natural \mathbb{C}^{N+2} valued 1-forms dZ_Λ on \mathfrak{B} are defined as differentials of the map $(Z_0, \dots, Z_{N+1}) \rightarrow Z_\Lambda$.

Recall from [CM74] and [W79] that a smoothly varying Q -frame $(Z_\Lambda) = (Z_\Lambda(p))$ for $p \in Q$ is said to be *adapted* to Q if $Z_0(p) = p$ as points of $\mathbb{C}\mathbb{P}^{N+1}$. A choice of Q -frame gives a smooth map from Q to \mathfrak{B} . It is shown in section 5 of [CM74] that if we use an adapted Q -frame to pull back the 1-forms π_Λ^Ω from \mathfrak{B} to Q and set

$$\theta := \frac{1}{2}\pi_0^{N+1}, \quad \theta^A := \pi_0^A, \quad \xi := -\pi_0^0 + \overline{\pi_0^0}, \quad (3.5)$$

we obtain a coframe (θ, θ^A) on Q and a form ξ satisfying the structure equation

$$d\theta = i\hat{l}_{A\bar{B}}\theta^A \wedge \theta^{\bar{B}} + \theta \wedge \xi.$$

In particular, it follows from (3.5) that the coframe $\{\theta^A, 2\theta\}$ is dual to the frame defined by $\{Z_A, Z_{N+1}\}$ on Q and hence depends only on the values of (Z_Λ) at the same points. Furthermore, there is a unique section $M \rightarrow Y$ for which the pullbacks of the forms $(\omega, \omega^\alpha, \phi)$ are $(\theta, \theta^\alpha, \xi)$ respectively. Then the pulled back forms $(\hat{\phi}_B^A, \hat{\phi}^A, \hat{\psi})$ are given by (5.8b) from [CM74]:

$$\hat{\phi}_B^A = \pi_B^A - \delta_B^A \pi_0^0, \quad \hat{\phi}^A = 2\pi_{\hat{n}+1}^A, \quad \hat{\psi} = -4\pi_{\hat{n}+1}^0. \quad (3.6)$$

As in (5.30) from [CM74], the pulled back forms π_Λ^Ω can be uniquely solved from (3.5) and (3.6):

$$\begin{aligned} (\hat{n} + 2)\pi_0^0 &= -\hat{\phi}_C^C - \xi, & \pi_0^A &= \theta^A, & \pi_0^{\hat{n}+1} &= 2\theta, \\ \pi_A^0 &= -i\hat{\phi}_A, & \pi_A^B &= \hat{\phi}_A^B + \delta_A^B \pi_0^0, & \pi_A^{\hat{n}+1} &= 2i\theta_A, \\ 4\pi_{\hat{n}+1}^0 &= -\hat{\psi}, & 2\pi_{\hat{n}+1}^A &= \hat{\phi}^A, & (\hat{n} + 2)\pi_{\hat{n}+1}^{\hat{n}+1} &= \hat{\phi}_D^{\bar{D}} + \xi. \end{aligned} \quad (3.7)$$

Thus, the pullback of π_Λ^Ω is completely determined by the pullbacks $(\theta, \theta^A, \xi, \hat{\phi}_B^A, \hat{\phi}^A, \hat{\psi})$. Following section 8 of [EHZ04], we note that for any choice of an admissible coframe (θ, θ^A) on Q , there exists an adapted Q -frame (Z_Λ) such that (3.7) holds with $\xi = 0$.

Chapter 4

Partial Rigidity for Degenerate Maps

In this chapter we prove our first result, Theorem 1.2.1, which we restate for the reader's convenience. We consider degenerate immersions of a Levi-nondegenerate hypersurface into a hyperquadric, where the signature of the target quadric may be strictly greater than that of the source manifold.

Theorem 1.2.1. *Let $M \subset \mathbb{C}^{n+1}$ be a smooth connected Levi-nondegenerate hypersurface of signature $l \leq n/2$ and $f : M \rightarrow Q_l^N$ a smooth CR mapping that is CR-transversal to Q_l^N at $f(p)$ for $p \in M$. Assume that f is constantly (k, s) -degenerate near p for some k and s . If $N - n - s < n$, then there is an open neighborhood V of p in M such that $f(V)$ is contained in the intersection of Q_l^N with a complex plane $P \subset \mathbb{C}^{N+1}$ of codimension s .*

4.1 Some Linear Algebra Lemmas

Lemma 4.1.1. *Let $\{L_1, \dots, L_d\}$ and $\{\tilde{L}_1, \dots, \tilde{L}_d\}$ be two bases for a d dimensional complex vector space with corresponding dual bases $\{\theta^1, \dots, \theta^d\}$ and $\{\tilde{\theta}^1, \dots, \tilde{\theta}^d\}$, and*

*suppose
$$\begin{pmatrix} \theta^1 \\ \vdots \\ \theta^d \end{pmatrix} = \begin{pmatrix} & & \\ & M & \\ & & \end{pmatrix} \begin{pmatrix} \tilde{\theta}^1 \\ \vdots \\ \tilde{\theta}^d \end{pmatrix}.$$
 If v and \tilde{v} denote the coordinates of a vector*

V in the L and \tilde{L} bases respectively, then $\tilde{v} = N^T v$ where $N = M^{-1}$.

Proof. We have $\theta^a = M_b^a \tilde{\theta}^b$, so we can check that $\tilde{L}_b = M_b^a L_a$. This implies that $L_a =$

$N_a^b \tilde{L}_b$. Then $V = v^a L_a = v^a N_a^b \tilde{L}_b$. Therefore we have $\tilde{v} = \begin{pmatrix} N_a^1 v^a \\ \vdots \\ N_a^d v^a \end{pmatrix} = N^T v$ as desired. \square

The following lemma will be a key ingredient in the proof of Theorem 1.2.1:

Lemma 4.1.2. *Let g be a diagonal matrix in \mathbb{C}^d with either positive or negative 1 in each diagonal entry and denote by $e_j = (0, \dots, 1, \dots, 0)^T$ the j^{th} standard basis vector in \mathbb{C}^d . Let E be the span of r independent vectors in \mathbb{C}^d , with $r + s = d$. Without loss of generality, suppose E is a graph over $\{e_{s+1}, \dots, e_d\}$, that is, there exists a $d \times r$ matrix of the form $\begin{pmatrix} C^T \\ I \end{pmatrix}$ where C^T is $s \times r$, whose columns span E . Then there exists an invertible matrix A in \mathbb{C}^d such that if $N = A^{-1}$, then for $v \in E$, $N^T v \in \text{span}\{e_{s+1}, \dots, e_d\}$ and if $\tilde{g} := A^* g A$, then $\tilde{g}_{pq} = 0$ when $p \in \{s+1, \dots, d\}$ and $q \in \{1, \dots, s\}$.*

Proof. Define I_1 and I_2 to be the $s \times s$ and $r \times r$ upper left and lower right blocks of g , respectively. Choose a matrix norm such that $\|I_j\| \leq 1$ for $j = 1, 2$ and nonzero constant λ such that $|\lambda|^2 > \max\{\|C^* I_2 C\|, \|I_2 C I_1 C^* I_2\|\}$.

We now show that $A := \begin{pmatrix} \lambda I & -\frac{1}{\lambda} I_1 C^* I_2 \\ C & I \end{pmatrix}$, where the upper left block is $s \times s$ and the lower right block is $r \times r$ satisfies the desired requirements. Note that by construction, A^T carries the span of $\{e_{s+1}, \dots, e_d\}$ to E , so N^T takes E to the span of $\{e_{s+1}, \dots, e_d\}$.

We compute $A^* g A$:

$$\begin{aligned} A^* g A &= \begin{pmatrix} \bar{\lambda} I & C^* \\ -\frac{1}{\lambda} I_2 C I_1 & I \end{pmatrix} \begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix} \begin{pmatrix} \lambda I & -\frac{1}{\lambda} I_1 C^* I_2 \\ C & I \end{pmatrix} \\ &= \begin{pmatrix} \bar{\lambda} I & C^* \\ -\frac{1}{\lambda} I_2 C I_1 & I \end{pmatrix} \begin{pmatrix} \lambda I_1 & -\frac{1}{\lambda} C^* I_2 \\ I_2 C & I_2 \end{pmatrix} \\ &= \begin{pmatrix} |\lambda|^2 (I_1 + \frac{1}{|\lambda|^2} C^* I_2 C) & 0 \\ 0 & I_2 + \frac{1}{|\lambda|^2} I_2 C I_1 C^* I_2 \end{pmatrix}. \end{aligned}$$

This shows that $A^* g A$ is block diagonal. To see that A is invertible, it suffices to show each block of $A^* g A$ is invertible. Up to a constant, each block is of the form $I_j + L$,

where L has norm less than 1 by our choice of λ . This implies that $I + I_j L$ is invertible (with the appropriate dimensions of I in each block), so there is a matrix D_j such that $(I + I_j L)D_j = I$. Hence by multiplying both sides on the left and right by I_j we have $(I_j + L)D_j I_j = I$, so $I_j + L$ is invertible, as desired. \square

4.2 Proof of Theorem 1.2.1

We choose an admissible coframe (θ, θ^A) on Q near $f(p)$ adapted to an admissible coframe (θ, θ^α) on M and denote by $(\omega_{\alpha\beta}^a)$ the second fundamental form of f relative to this coframe. Since the mapping f is (k, s) -degenerate near p , we have that the dimension of $\text{span}\{\omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_t}^a, 2 \leq t \leq k\}$ is $r = d - s$ near p , where $d = N - n$. We introduce some notation; the indices $*, \#$ run over the set $n + 1, \dots, n + r$ (possibly empty) and the indices i, j run over the set $n + r + 1, \dots, N$.

We now fix α, β and identify $(\omega_{\alpha\beta}^a(p))$ as a vector in \mathbb{C}^{N-n} . We apply Lemma 4.1.2 with $g_{a\bar{b}}$ as the matrix g and after the above identification (and corresponding identifications for the covariant derivatives), we let $E = \text{span}\{\omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_t}^a, 2 \leq t \leq k\}$, a subspace of \mathbb{C}^{N-n} . The lemma gives a smooth matrix-valued function A , and we change

basis on the normal bundle via
$$\begin{pmatrix} \theta^{n+1} \\ \vdots \\ \theta^N \end{pmatrix} = \begin{pmatrix} & \\ & A \end{pmatrix} \begin{pmatrix} \tilde{\theta}^{n+1} \\ \vdots \\ \tilde{\theta}^N \end{pmatrix}$$
 then by Lemma 4.1.1 we have

$$\text{span}\{\omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_t}^\# L_\#, 2 \leq t \leq k\} = \text{span}\{\tilde{L}_\#\}, \text{ and } \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_t}^j \equiv 0, \quad t \geq 2. \quad (4.1)$$

To obtain (4.1) we have also changed basis to switch $\tilde{L}_\#$ and \tilde{L}_j . The reader may verify that Hermitian conjugation by such a change of basis matrix preserves the property of being block diagonal.

We now relabel and omit the tilde notation. Note that the matrix of the Levi form on the normal space is no longer necessarily constant, but does satisfy at each point the conclusion of Lemma 4.1.2, so $g_{\#\bar{j}} = 0$. Also, we still have the relations $f^*(\theta^a) = 0$ and $\hat{g}_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}}$. Note that the inverse of a block diagonal matrix is block diagonal, so $g^{A\bar{B}}$ has the same form as $g_{A\bar{B}}$.

Because $\hat{\omega}_\#^j$ is a 1-form on M , we have

$$\hat{\omega}_\#^j = \hat{\omega}_\#^j{}_\mu \theta^\mu + \hat{\omega}_\#^j{}_{\bar{\nu}} \theta^{\bar{\nu}} + \hat{\omega}_\#^j{}_0 \theta \quad (4.2)$$

for coefficients $\hat{\omega}_\#^j{}_\mu$, $\hat{\omega}_\#^j{}_{\bar{\nu}}$, and $\hat{\omega}_\#^j{}_0$.

Now by the definition of covariant derivative, we have

$$\begin{aligned} \nabla \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_t}^j &= d\omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_t}^j + \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_t}^i \hat{\omega}_i^j + \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_t}^\# \hat{\omega}_\#^j \\ &\quad - \sum_{q=1}^t \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_{q-1} \mu \gamma_{q+1} \dots \gamma_t}^j \hat{\omega}_{\gamma_q}^\mu \end{aligned}$$

so by (4.1) we have

$$\nabla \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_t}^j = \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_t}^\# \hat{\omega}_\#^j.$$

This implies that

$$\omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_t \mu}^j = \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_t}^\# \hat{\omega}_\#^j{}_\mu$$

and because the left side is zero we have

$$\omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_t}^\# \hat{\omega}_\#^j{}_\mu = 0. \quad (4.3)$$

Now if j, μ are fixed and $\hat{\omega}_\#^j{}_\mu \neq 0$ for some $\#$ then pick r independent vectors with r components $(\omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_t}^*)$, make a matrix B with these as the rows and let v be the vector $(\hat{\omega}_\#^j{}_\mu)$ as $\#$ varies. Then $Bv = 0$ contradicting independence of the rows of B .

This implies that

$$\hat{\omega}_\#^j{}_\mu = 0. \quad (4.4)$$

Now applying Proposition 3.2.1, and noting that, by equations (2.9) and (4.1) we have $\hat{\omega}_\alpha^j = 0$ and $\tau^a = 0$, we find

$$\hat{\phi}_\alpha^j = \hat{D}_\alpha^j \theta, \quad \hat{\phi}^j = \hat{D}_\mu^j \theta^\mu + \hat{E}^j \theta, \quad (4.5)$$

and

$$\hat{\phi}_\alpha^\# = \hat{\omega}_\alpha^\# + \hat{D}_\alpha^\# \theta, \quad \hat{\phi}^\# = \hat{D}_\mu^\# \theta^\mu + \hat{E}^\# \theta. \quad (4.6)$$

Next, we differentiate $\hat{\phi}_\alpha^j$ and compute mod θ to obtain

$$d\hat{\phi}_\alpha^j \equiv i\hat{D}_\alpha^j g_{\mu\bar{\nu}} \theta^\mu \wedge \theta^{\bar{\nu}} \quad \text{mod } \theta$$

On the other hand, we may compute $d\hat{\phi}_\alpha^j \bmod \theta$ using the structure equations (3.1). We have

$$\begin{aligned} d\hat{\phi}_\alpha^j &\equiv \hat{\phi}_\alpha^A \wedge \hat{\phi}_A^j + i\theta_\alpha \wedge \hat{\phi}^j - i\phi_\alpha \wedge \theta^j - i\delta_\alpha^j \phi_A \wedge \theta^A - \frac{\delta_\alpha^j}{2} \psi \wedge \theta + \Phi_\alpha^j \\ &\equiv \hat{\phi}_\alpha^A \wedge \hat{\phi}_A^j + i\theta_\alpha \wedge \hat{\phi}^j \quad \bmod \theta. \end{aligned}$$

We note that in the structure equation above the third term is zero because the pullback of θ^j vanishes, the fourth and fifth terms are zero because of the indices of the kronecker delta, and the last term is zero because of the vanishing pseudoconformal curvature of the target hyperquadric.

We expand the above to obtain

$$\begin{aligned} d\hat{\phi}_\alpha^j &\equiv \hat{\phi}_\alpha^\beta \wedge \hat{\phi}_\beta^j + \hat{\phi}_\alpha^\# \wedge \hat{\phi}_\#^j + \hat{\phi}_\alpha^i \wedge \hat{\phi}_i^j + i\theta_\alpha \wedge \hat{\phi}^j \\ &\equiv \hat{\phi}_\alpha^\# \wedge \hat{\phi}_\#^j + i\theta_\alpha \wedge \hat{\phi}^j \\ &\equiv \hat{\omega}_{\alpha\ \mu}^\# \theta^\mu \wedge \hat{\phi}_\#^j - i\hat{\phi}^j \wedge g_{\alpha\bar{A}} \theta^{\bar{A}} \quad \bmod \theta. \end{aligned}$$

In the second equivalence we used equation (4.5) and computed $\bmod \theta$, and in the last equivalence we used (4.5), (4.6), and equation (2.9).

Now we combine the two equations for $d\hat{\phi}_\alpha^j$ and group terms to obtain

$$\hat{\omega}_{\alpha\ \mu}^\# \theta^\mu \wedge \hat{\phi}_\#^j \equiv i(g_{\alpha\bar{\nu}} \hat{D}_\mu^j + g_{\mu\bar{\nu}} \hat{D}_\alpha^j) \theta^\mu \wedge \theta^{\bar{\nu}} \quad \bmod \theta. \quad (4.7)$$

By Proposition 3.2.1 and equation (4.2), we compute $\hat{\phi}_\#^j$ and identify the coefficients of $\theta^\mu \wedge \theta^{\bar{\nu}}$ to obtain

$$\hat{\omega}_{\alpha\ \mu}^\# \hat{\omega}_\#^j_{\bar{\nu}} = i(g_{\alpha\bar{\nu}} \hat{D}_\mu^j + g_{\mu\bar{\nu}} \hat{D}_\alpha^j).$$

This holds in a neighborhood of p , so we now work at a point q close to p . Let

$$f_\#(z) = \hat{\omega}_{\alpha\ \mu}^\# z^\alpha z^\mu \quad \text{and} \quad g_\#(z) = \hat{\omega}_\#^j_{\bar{\nu}} z^{\bar{\nu}}.$$

Then we have that

$$\begin{aligned} \sum_\# f_\#(z) \overline{g_\#(z)} &= \hat{\omega}_{\alpha\ \mu}^\# \hat{\omega}_\#^j_{\bar{\nu}} z^\alpha z^\mu z^{\bar{\nu}} \\ &= i(g_{\alpha\bar{\nu}} \hat{D}_\mu^j + g_{\mu\bar{\nu}} \hat{D}_\alpha^j) z^\alpha z^\mu z^{\bar{\nu}} \\ &= \langle z, z \rangle_g (i\hat{D}_\mu^j z^\mu + i\hat{D}_\alpha^j z^\alpha). \end{aligned}$$

Therefore by Lemma 2.5.1, since $\#$ runs over an index set of size r and by assumption $r = N - n - s < n$, we have

$$\hat{\omega}_{\alpha\ \mu}^{\#} \hat{\omega}_{\# \bar{\nu}}^j = 0. \quad (4.8)$$

This implies that $g_{\alpha\bar{\nu}} \hat{D}_{\mu}^j + g_{\mu\bar{\nu}} \hat{D}_{\alpha}^j = 0$. Let $\alpha = \mu$ and choose $\bar{\nu}$ such that $g_{\alpha\bar{\nu}} \neq 0$, which exists since no row is completely zero. This implies $\hat{D}_{\alpha}^j = 0$, so

$$\hat{\phi}_{\alpha}^j = 0, \quad \hat{\phi}^j = \hat{E}^j \theta. \quad (4.9)$$

Combining the structure equation for $d\hat{\phi}_{\alpha}^j$ with the above result yields

$$0 = \hat{\phi}_{\alpha}^A \wedge \hat{\phi}_A^j + i\theta_{\alpha} \wedge \hat{\phi}^j - i\hat{\phi}_{\alpha} \wedge \hat{\theta}^j.$$

We only consider those terms containing a $\theta^{\mu} \wedge \theta$ and discover, using Proposition 3.2.1 and equation (4.2), that

$$\begin{aligned} 0 &= \hat{\phi}_{\alpha}^{\#} \wedge \hat{\phi}_{\#}^j \\ &= (\omega_{\alpha\ \mu}^{\#} \theta^{\mu} + \hat{D}_{\alpha}^{\#} \theta) \wedge (\hat{\omega}_{\# \mu}^j \theta^{\mu} + \hat{\omega}_{\# \bar{\nu}}^j \theta^{\bar{\nu}} + (\hat{\omega}_{\# 0}^j + \hat{D}_{\#}^j) \theta). \end{aligned}$$

Keeping the $\theta^{\mu} \wedge \theta$ terms and using equation (4.4), we obtain

$$0 = \omega_{\alpha\ \mu}^{\#} (\hat{\omega}_{\# 0}^j + \hat{D}_{\#}^j). \quad (4.10)$$

Now we would like to show that $\hat{\phi}_{\#}^j = 0$, so since $\hat{\phi}_{\#}^j = \hat{\omega}_{\# \bar{\nu}}^j \theta^{\bar{\nu}} + (\hat{\omega}_{\# 0}^j + \hat{D}_{\#}^j) \theta$ by Proposition 3.2.1 and equation (4.4), it suffices to show

$$\omega_{\gamma_1 \gamma_2 \gamma_3 \dots \gamma_t}^{\#} \hat{\omega}_{\# \bar{\nu}}^j = \omega_{\gamma_1 \gamma_2 \gamma_3 \dots \gamma_t}^{\#} (\hat{\omega}_{\# 0}^j + \hat{D}_{\#}^j) = 0, \quad t \geq 2 \quad (4.11)$$

by the same reason equation (4.3) implied (4.4).

Before proving (4.11), we first wish to show that $\hat{\omega}_{\# \bar{\nu}; \mu}^j$ is a sum of multiples of the Levi form. We differentiate the expression for $\hat{\phi}_{\#}^j$ in Proposition 3.2.1, set it equal to the corresponding structure equation, and compute mod θ to obtain

$$\hat{\phi}_{\#}^A \wedge \hat{\phi}_A^j \equiv d\omega_{\#}^j + iD_{\#}^j g_{\mu\bar{\nu}} \theta^{\mu} \wedge \theta^{\bar{\nu}}.$$

We use equation (4.9) and Proposition 3.2.1 to simplify the left side and equations (4.2) and (4.4) to simplify the right side mod θ . This yields

$$\hat{\omega}_{\#}^a \wedge \hat{\omega}_a^j \equiv d\omega_{\# \bar{\nu}}^j \wedge \theta^{\bar{\nu}} + \hat{\omega}_{\# \bar{\nu}}^j d\theta^{\bar{\nu}} + i(\hat{\omega}_{\# 0}^j g_{\mu\bar{\nu}} + D_{\#}^j g_{\mu\bar{\nu}}) \theta^{\mu} \wedge \theta^{\bar{\nu}}.$$

We now only consider terms involving $\theta^\mu \wedge \theta^{\bar{\nu}}$. Hence we have

$$(\hat{\omega}_{\# \mu}^a \hat{\omega}_a^j - \hat{\omega}_a^j \hat{\omega}_{\# \bar{\nu}}^a) \theta^\mu \wedge \theta^{\bar{\nu}} \equiv d\omega_{\# \bar{\nu}}^j \wedge \theta^{\bar{\nu}} + \hat{\omega}_{\# \bar{\nu}}^j d\theta^{\bar{\nu}} + (\hat{\omega}_{\# 0}^j g_{\mu \bar{\nu}} + D_{\#}^j g_{\mu \bar{\nu}}) \theta^\mu \wedge \theta^{\bar{\nu}}.$$

After using the structure equation from (3.1) for $d\theta^{\bar{\nu}}$, Proposition 3.2.1, and simplifying, we note that $d\theta^{\bar{\alpha}} \equiv -\omega_{\bar{\nu}}^{\bar{\alpha}} \wedge \theta^{\bar{\nu}} \pmod{\theta}$, so the coefficient of $\theta^\mu \wedge \theta^{\bar{\nu}}$ in the expression $\hat{\omega}_{\# \bar{\alpha}}^j d\theta^{\bar{\alpha}}$ is $-\hat{\omega}_{\# \bar{\alpha}}^j \hat{\omega}_{\bar{\nu}}^{\bar{\alpha}}$. Hence we are left with the equality

$$(d\omega_{\# \bar{\nu}}^j)_{\mu} - \hat{\omega}_a^j \hat{\omega}_{\# \mu}^a + \hat{\omega}_{\# \bar{\nu}}^a \hat{\omega}_a^j - \hat{\omega}_{\# \bar{\alpha}}^j \hat{\omega}_{\bar{\nu}}^{\bar{\alpha}} = -(\hat{\omega}_{\# 0}^j g_{\mu \bar{\nu}} + D_{\#}^j g_{\mu \bar{\nu}}).$$

However, the left hand side equals $\hat{\omega}_{\# \bar{\nu}; \mu}^j$, so $\hat{\omega}_{\# \bar{\nu}; \mu}^j$ is a sum of multiples of the Levi form. We now covariantly differentiate equation (4.8) and recall that $\nabla g_{\mu \bar{\nu}} = 0$ to obtain that $\omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_l}^{\#} \hat{\omega}_{\# \bar{\nu}}^j$ is a sum of multiples of the Levi form, so by using Lemma 2.5.1 as in the derivation of (4.8), we conclude $\omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_l}^{\#} \hat{\omega}_{\# \bar{\nu}}^j = 0$. This proves that the first expression in (4.11) vanishes.

Now we work modulo $\theta \wedge \theta^{\bar{\beta}}$ and $\theta^\alpha \wedge \theta^{\bar{\beta}}$. Since $\hat{\phi}_{\#}^j = \omega_{\#}^j + D_{\#}^j \theta$, we have $d\hat{\phi}_{\#}^j \equiv d\omega_{\#}^j + dD_{\#}^j \wedge \theta$. On the other hand, we use the structure equations (3.1) and simplify, yielding the identity

$$\hat{\phi}_{\#}^a \wedge \hat{\phi}_a^j \equiv d(\hat{\omega}_{\# 0}^j + D_{\#}^j) \wedge \theta.$$

We rewrite the left hand side of the above equation using Proposition 3.2.1, simplify, and collect coefficients of $\theta^\mu \wedge \theta$. This gives

$$\partial_{\mu}(\hat{\omega}_{\# 0}^j + D_{\#}^j) + \omega_a^j (\hat{\omega}_{\# 0}^a + D_{\#}^a) - \omega_{\# \mu}^a (\hat{\omega}_a^j + D_a^j) = 0$$

which implies that $(\hat{\omega}_{\# 0}^j + D_{\#}^j)_{; \mu}$ is zero. Therefore all higher order covariant derivatives in the directions $\theta^\alpha, \theta^{\bar{\beta}}$ are zero, so by Lemma 2.5.1, this implies that the second expression in equation (4.11) vanishes. Hence we now have that $\hat{\phi}_{\#}^j = 0$.

Since $\phi_{\alpha}^j = 0$, we examine $d\phi_{\alpha}^j$ and use the structure equation and our previous results to obtain

$$\begin{aligned} 0 &= \phi_{\alpha}^A \wedge \phi_A^j + i\theta_{\alpha} \wedge \phi^j \\ &= i\theta_{\alpha} \wedge (\hat{E}^j \theta). \end{aligned}$$

This implies that $\hat{E}^j = 0$, so $\hat{\phi}^j = 0$ also.

So far we have shown that $\hat{\phi}_\alpha^j = \hat{\phi}_\#^j = \hat{\phi}^j = 0$. We choose an adapted \mathcal{Q} -frame (Z_Λ) on Q near $f(p)$. We can choose (Z_Λ) corresponding to our coframe (θ, θ^A) such that the following relations are satisfied (see the second row of equation (3.7)).

$$\Pi_A^0 = -i\hat{\phi}_A, \quad \Pi_A^B = \hat{\phi}_A^B + \delta_A^B \Pi_0^0, \quad \Pi_A^{\hat{n}+1} = 2i\theta_A.$$

First, note that

$$\Pi_j^{\hat{n}+1} = 2i\theta_j = 2i\theta^{\bar{A}} g_{\bar{A}j} = 0$$

because $\theta^{\bar{a}} = 0$ on M and $g_{\bar{\alpha}j} = 0$.

Next, we see that

$$\Pi_j^0 = -i\hat{\phi}_j = -i\hat{\phi}^{\bar{A}} g_{\bar{A}j} = -i\hat{\phi}^{\bar{\alpha}} g_{\bar{\alpha}j} - i\hat{\phi}^{\bar{\#}} g_{\bar{\#}j} - i\hat{\phi}^{\bar{i}} g_{\bar{i}j}.$$

The first term in the above sum is zero because of the indices of the Levi form. The second term is zero again because of the indices of the Levi form, due to our change of basis at the beginning of the proof. The third term is zero because $\hat{\phi}^{\bar{i}} = 0$.

Now we analyze Π_j^α , noting that $\delta_j^\alpha = 0$ and using the symmetry relation $\hat{\phi}_{j\bar{\beta}} = -\hat{\phi}_{\bar{\beta}j}$. We have

$$\Pi_j^\alpha = \hat{\phi}_j^\alpha = \hat{\phi}_{j\bar{\beta}} g^{\bar{\beta}\alpha} = -\hat{\phi}_{\bar{\beta}j} g^{\bar{\beta}\alpha} = -\hat{\phi}_{\bar{\beta}}^{\bar{A}} g_{\bar{A}j} g^{\bar{\beta}\alpha} = 0$$

because $g_{\bar{A}j} = 0$ unless A is in the range of j , and then $\hat{\phi}_{\bar{\alpha}}^{\bar{j}} = 0$.

We perform a similar analysis of $\Pi_j^\#$.

$$\Pi_j^\# = \hat{\phi}_j^\# = -\hat{\phi}_{\bar{A}}^{\bar{B}} g_{\bar{B}j} g^{\bar{A}\#} = 0,$$

because $g_{\bar{B}j} = 0$ unless B is in the range of j , $g^{\bar{A}\#} = 0$ unless A is in the range of $\#$, and if both of these cases occur, then $\hat{\phi}_{\bar{\#}}^{\bar{j}} = 0$.

This shows that $\Pi_j^\Omega = 0$ unless $\Omega \in \{n+r+1, \dots, N\}$. Therefore, since the Maurer-Cartan forms are defined by $dZ_\Lambda = \Pi_\Lambda^\Omega Z_\Omega$, we have

$$dZ_i = \Pi_i^j Z_j, \tag{4.12}$$

expressing that the derivatives of the vectors Z_i are linear combinations of Z_j at each point.

We now claim that the span of $Z_j = Z_j(q)$ must be constant in the Grassmannian of \mathbb{C}^{N+2} for $q \in M$ near p . To prove this, let $\gamma(t)$ be a smooth curve in $f(M)$ connecting points p and q , with $\gamma(0) = p$. By composing γ with the map taking a point in \mathbb{C}^{N+2} to the chosen Q -frame at that point, we may pull back equation (4.12) to the real line, yielding a system of ordinary differential equations. If we write $Z_i(t) = Z_i \circ \gamma(t)$ and relabel i, j , so they are between 1 and s , then letting $Y(t)$ be the $(N+2) \times s$ matrix with columns $Z_j(t)$, the system of differential equations is of the form

$$\dot{Y}(t) = Y(t)f(t) \tag{4.13}$$

for some $s \times s$ matrix of smooth functions $f(t)$. Note that for t sufficiently close to 0, the matrix $Y(t)$ has full rank at each point (and hence a left inverse at $t = 0$).

We would now like to show the existence of an $s \times s$ matrix of functions $\phi(t)$ such that $Y(t) = Y(0)\phi(t)$. This would imply that, as t varies, all curves in \mathbb{C}^{N+2} given by the columns of Y would lie in the plane spanned by the columns of $Y(0)$, and thus that the span of all Z_j are constant in the Grassmannian. We notice that if such a matrix $\phi(t)$ exists, then by substituting into equation (4.13), we must have $Y(0)\dot{\phi}(t) = Y(0)\phi(t)f(t)$, and by multiplying by a left inverse of $Y(0)$, we obtain

$$\dot{\phi}(t) = \phi(t)f(t).$$

Considering this system as a vector of equations, we may apply the standard theorem on existence and uniqueness of ordinary differential equations (see, for instance Theorem 17.9 in [Le02]) with initial condition $\phi(0) = Id$ to obtain the desired solution.

Since the span of the Z_j are constant in the Grassmannian of \mathbb{C}^{N+2} for $q \in M$ near p , the vectors $Z_0(q)$ lie in the constant $(N+2-s)$ -dimensional orthogonal subspace (with respect to the inner product (3.3)). Since, by definition of the adapted Q -frame, $Z_0(q)$ gives the reference point $q \in \mathbb{C}\mathbb{P}^{N+1}$, we conclude that $f(q)$ is contained in a fixed $(N+1-s)$ -dimensional plane P in \mathbb{C}^{N+1} for $q \in M$ near p .

Chapter 5

Dimensions of E_k for Embeddings

This chapter is devoted to the proof of our main technical result, Theorem 5.0.1 below. This result relates the dimensions of the E_k spaces for two embeddings. To simplify notation, we write ω_α^a where $a \in \{1, \dots, N-n\}$ rather than ω_α^{a+n} for the second fundamental forms of the mappings. The proof is given in section 5.3.

Theorem 5.0.1. *Let $M \subseteq \mathbb{C}^{n+1}$ be a smooth Levi-nondegenerate hypersurface of signature $l \leq n/2$ and $p \in M$. Let $f_0 : M \rightarrow Q_l^{N_0}$ and $f : M \rightarrow Q_{l'}^N$ be smooth CR mappings that are CR transversal to $Q_l^{N_0}$ at $f_0(p)$ and $Q_{l'}^N$ at $f(p)$, respectively, and $N_0 \leq N$. Fix an admissible coframe (θ, θ^α) on M and choose corresponding coframes $(\hat{\theta}, \hat{\theta}^A)_{A=1, \dots, N_0}$ and $(\hat{\theta}, \hat{\theta}^A)_{A=1, \dots, N}$ on $Q_l^{N_0}$ and $Q_{l'}^N$ adapted to $f_0(M)$ and $f(M)$, respectively. Let $(\hat{\omega}_{\gamma_1}^a)_{a=1, \dots, N_0-n}$ and $(\omega_{\gamma_1}^a)_{a=1, \dots, N-n}$ denote the second fundamental forms of f_0 and f , respectively, relative to these coframes. Let $k \geq 2$ be an integer and assume that the spaces $\hat{E}_j(q)$ and $E_j(q)$ for $2 \leq j \leq k$, are of constant dimension for q near p . Then for each k ,*

(a) *If $l = n/2$ or $l' = N/2$ or f is side preserving, and if either $(N_0 - n) + (l' - l) < l$ or $(N - l') - (n - l) < l$, we have $\dim(E_k) \leq \dim(\hat{E}_k) + \min(l' - l, (N - l') - (n - l))$.*

(b) *If f is side reversing and if $l' < n$, we have $\dim(E_k) \leq \dim(\hat{E}_k) + l' - (n - l)$.*

5.1 Some Useful Lemmas

A key ingredient in the proof of Theorem 5.0.1 is the Gauss equation for the second fundamental form of the embedding. A more general and precise version is stated and proved in [EHZ04] where it appears as Theorem 2.3. The statement here is the same as Lemma 4.3 in [BEH08].

Lemma 5.1.1. *Let $M \subset \mathbb{C}^{n+1}$ be a smooth Levi-nondegenerate hypersurface of signature $l \leq \frac{n}{2}$, $f : M \rightarrow Q_{l'}^N \subset \mathbb{C}^{N+1}$ a smooth CR mapping that is CR transversal to $Q_{l'}^N$ along M , $\omega_{\alpha\beta}^a$ its second fundamental form, and $l \leq l'$. Then,*

$$0 = S_{\alpha\bar{\beta}\mu\bar{\nu}} + g_{a\bar{b}} \omega_{\alpha\mu}^a \omega_{\bar{\beta}\bar{\nu}}^{\bar{b}} + T_{\alpha\bar{\beta}\mu\bar{\nu}},$$

where $S_{\alpha\bar{\beta}\mu\bar{\nu}}$ is the Chern-Moser pseudoconformal curvature of M and $T_{\alpha\bar{\beta}\mu\bar{\nu}}$ is a conformally flat tensor.

We shall need the following lemma regarding conformal flatness of certain covariant derivatives of the second fundamental form. This lemma appears with proof as Lemma 4.1 in [BEH08].

Lemma 5.1.2. *Let M , f , and $\omega_{\alpha\beta}^a$ be as in Lemma 5.1.1. Then the covariant derivative tensor $\omega_{\alpha\beta;\bar{\gamma}}^a$ is conformally flat.*

The following linear algebra Lemma will be useful

Lemma 5.1.3. *Let $\{w_1, \dots, w_m\}$ and $\{v_1, \dots, v_m\}$ be vectors in \mathbb{C}^n such that $\langle w_i, w_j \rangle' = \langle v_i, v_j \rangle$. Here $\langle x, y \rangle$ denotes the standard inner product on \mathbb{C}^n and $\langle x, y \rangle' = y^* I_k x$, where I_k is the $n \times n$ diagonal matrix with first k entries equal to -1 and remaining $n - k$ entries equal to 1. Let $W = \text{span}\{w_1, \dots, w_m\}$ and $V = \text{span}\{v_1, \dots, v_m\}$, then $\dim(W) \leq \dim(V) + \min(k, n - k)$.*

Proof. Let w_{i_1}, \dots, w_{i_k} be a basis for W and define a linear map ϕ from W to V by $\phi(w_{i_j}) = v_{i_j}$. Suppose $x = \sum_{r=1}^k a_r w_{i_r}$ is in the kernel of ϕ , so $\sum_{r=1}^k a_r v_{i_r} = 0$. Let a denote the coordinate vector of x , and $(\langle w_{i_r}, w_{i_s} \rangle')$, and $(\langle v_{i_r}, v_{i_s} \rangle)$ be square matrices of size $\dim(W)$ whose (r, s) entries are respectively $\langle w_{i_r}, w_{i_s} \rangle'$ and $\langle v_{i_r}, v_{i_s} \rangle$. Then we have $\langle x, x \rangle' = a^* (\langle w_{i_r}, w_{i_s} \rangle') a = a^* (\langle v_{i_r}, v_{i_s} \rangle) a = \langle \phi(x), \phi(x) \rangle = 0$. This shows that the kernel of ϕ is an isotropic subspace of \mathbb{C}^n with respect to $\langle \cdot, \cdot \rangle'$.

We may write $\mathbb{C}^n = S_k \oplus S_{n-k}$, the direct sum of two orthogonal subspaces, where $S_k = \text{span}\{e_1, \dots, e_k\}$, the span of the first k standard basis vectors, and $S_{n-k} = \text{span}\{e_{n-k+1}, \dots, e_n\}$. Define a map $\pi_k : \ker(\phi) \rightarrow S_k$ by projection, and suppose $\pi_k(v) = 0$. Then $v \in S_{n-k}$, but since $\langle v, v \rangle' = 0$ and this Hermitian product is definite on S_{n-k} , v must be 0. Hence π_k is an injection, so the dimension of $\ker(\phi) \leq k$. Similarly one can see that the dimension of $\ker(\phi) \leq n - k$, so the dimension is less than their minimum of k and $n - k$. By standard linear algebra, we observe that $\dim(W) = \dim(\text{image}(\phi)) + \dim(\ker(\phi)) \leq \dim(V) + \min(k, n - k)$, as desired. \square

5.2 Commuting Covariant Derivatives

It will be necessary to know how covariant derivatives of the second fundamental form commute. Given a CR embedding $f : M \rightarrow \hat{M}$, we now recall some facts about the pseudoconformal connection on \hat{M} pulled back to M . Suppose (θ, θ^A) is an adapted coframe for the pair (M, \hat{M}) . We use the same notation as in the Preliminaries section. We denote with $\hat{\cdot}$ the pseudoconformal connection forms on \hat{M} pulled back to M , where the indices run from 1 to N . Recall that $(\omega, \omega^\alpha, \omega^{\bar{\alpha}}) = (\hat{\omega}, \hat{\omega}^\alpha, \hat{\omega}^{\bar{\alpha}}) = (\theta, \theta^\alpha, \theta^{\bar{\alpha}})$ and $\hat{\omega}^\alpha = 0$ on M . We do not expect $(\phi_\beta^\alpha, \phi^\alpha, \psi)$ and $(\hat{\phi}_\beta^\alpha, \hat{\phi}^\alpha, \hat{\psi})$ to be equal, but since $\hat{\omega}_\beta^\alpha = \omega_\beta^\alpha$ and $\hat{\tau}^\alpha = \tau^\alpha$, Proposition 3.2.1 implies

$$\hat{\phi}_\beta^\alpha = \phi_\beta^\alpha + C_\beta^\alpha \theta, \quad \hat{\phi}^\alpha = \phi^\alpha + C_\mu^\alpha \theta^\mu + F^\alpha \theta, \quad \hat{\psi} = \psi + iF_\mu \theta^\mu - iF_{\bar{\nu}} \theta^{\bar{\nu}} + A\theta \quad (5.1)$$

where

$$C_\beta^\alpha := \hat{D}_\beta^\alpha - D_\beta^\alpha, \quad F^\alpha := \hat{E}^\alpha - E^\alpha, \quad A := \hat{B} - B$$

and $\hat{D}_\beta^\alpha, \hat{E}^\alpha, \hat{B}$ are the analogues for \hat{M} of the functions from Proposition 3.2.1, restricted to M . We also record the following expression for $C_{\alpha\bar{\beta}}$ which appears as equation (6.8) in [EHZ04].

$$C_{\alpha\bar{\beta}} = \frac{i(\hat{S}_a^\alpha{}_{\alpha\bar{\beta}} + \omega_\mu^a{}_\alpha \omega^{\mu a\bar{\beta}})}{n+2} - \frac{i(\hat{S}_a^\alpha{}_\mu + \omega_\mu^a{}_\nu \omega^{\mu a\nu})g_{\alpha\bar{\beta}}}{2(n+1)(n+2)}. \quad (5.2)$$

The following is a more specific version of Lemma 4.2 in [BEH08], where we give an explicit formula for the part which is not conformally flat.

Lemma 5.2.1. *Let M , f , and $\omega_{\alpha\beta}^a$ be as in Lemma 5.1.1, and $p \in M$. Then for any $s \geq 2$, we have*

$$\omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_s \alpha \bar{\beta}}^a - \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_s \bar{\beta} \alpha}^a \equiv \sum_{j=1}^s d_{\alpha \bar{\beta}}(\omega_{\gamma_j}^\mu) \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_{j-1} \mu \gamma_{j+1} \dots \gamma_s}^a - C_{\alpha \bar{\beta} c}^a \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_s}^c \quad (5.3)$$

where equivalence is modulo a conformally flat tensor, $d_{\alpha \bar{\beta}}(\omega_{\gamma_j}^\mu)$ is the coefficient of $\theta^\alpha \wedge \theta^{\bar{\beta}}$ in $d\omega_{\gamma_j}^\mu$, and $C_{\alpha \bar{\beta} c}^a$ is given by

$$C_{\alpha \bar{\beta} c}^a \equiv \omega_{\alpha}^{\bar{\beta} a} \omega_{\bar{\beta} c \bar{\beta}} + i \delta_c^a \hat{D}_{\bar{\beta} \alpha}.$$

Proof. We use the pseudoconformal connections introduced in chapters 2 and 3. We observe that the left hand side of (5.3) is a tensor, hence it is enough to show (5.3) at each fixed $p \in M$ with respect to any choice of adapted coframe near p . By making a unitary change of coframe $\theta^\alpha \rightarrow u_b^\alpha \theta^b$ and $\theta^a \rightarrow u_b^a \theta^b$ in the tangential and normal directions, we may choose an adapted coframe near p such that $\omega_\alpha^{\bar{\beta}}(p) = \omega_a^b(p) = 0$ (c.f. Lemma 2.1 in [Le88]).

Using (2.10), we calculate that at p ,

$$\begin{aligned} \nabla \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_s \alpha}^a &= d\omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_s \alpha}^a + \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_s \alpha}^b \omega_b^a - \sum_{j=1}^{s+1} \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_{j-1} \mu \gamma_{j+1} \dots \gamma_s \alpha}^a \omega_{\gamma_j}^\mu \\ &= d\omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_s \alpha}^a \end{aligned}$$

by our choice of coframe above. Substitution yields

$$\begin{aligned} \nabla \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_s \alpha}^a &= d(d\omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_s \alpha}^a + \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_s \alpha}^b \omega_b^a - \sum_{j=1}^s \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_{j-1} \mu \gamma_{j+1} \dots \gamma_s \alpha}^a \omega_{\gamma_j}^\mu) \\ &= \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_s \alpha}^b d\omega_b^a - \sum_{j=1}^s \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_{j-1} \mu \gamma_{j+1} \dots \gamma_s \alpha}^a d(\omega_{\gamma_j}^\mu). \end{aligned} \quad (5.4)$$

Proposition 3.2.1 implies that

$$d\hat{\phi}_b^a = d\omega_b^a + d(\hat{D}_b^a \theta). \quad (5.5)$$

Since $d\theta$ is conformally flat, we see that $d\hat{\phi}_b^a \equiv d\omega_b^a + d\hat{D}_b^a \wedge \theta$, and the coefficient of $\theta^\alpha \wedge \theta^{\bar{\beta}}$ of $d\hat{\phi}_b^a$ equals that of $d\omega_b^a$.

Using (5.4) and (5.5), we see that by the definition of the wedge product in terms of alternating product, the left hand side of (5.3) at p is equivalent, modulo a conformally

flat tensor, to the coefficient in front of $\theta^\alpha \wedge \theta^{\bar{\beta}}$ in the expression

$$\sum_{j=1}^s d\omega_{\gamma_j}^\mu \omega_{\gamma_1 \gamma_2 \gamma_3 \dots \gamma_{j-1} \mu \gamma_{j+1} \dots \gamma_s}^a - \omega_{\gamma_1 \gamma_2 \gamma_3 \dots \gamma_s}^c d\hat{\phi}_c^a.$$

Hence we would like to show that the coefficient in front of $\theta^\alpha \wedge \theta^{\bar{\beta}}$ in $d\hat{\phi}_c^a$ has the form of the $C_{\alpha\bar{\beta}c}^a$ given in the statement of the Lemma.

Note that we may work mod θ because we are only looking for the coefficient in front of $\theta^\alpha \wedge \theta^{\bar{\beta}}$. The structure equations (3.1) give

$$d\phi_c^a = \phi_c^B \wedge \phi_B^a + i\theta_c \wedge \phi^a - i\phi_c \wedge \theta^a - i\delta_c^a \phi_A \wedge \theta^A - \frac{\delta_c^a}{2} \psi \wedge \theta + \Phi_c^a. \quad (5.6)$$

Since $\phi_b^a \equiv \omega_b^a = 0$ by our choice of coframe at p , the first time in the right side of (5.6) becomes $\hat{\phi}_c^\rho \wedge \hat{\phi}_\rho^a$. Also, $\theta_c = g_{c\bar{B}}\theta^{\bar{B}}$, so if $1 \leq B \leq n$, $g_{c\bar{B}} = 0$, and if $n+1 \leq B \leq N$, $\theta^{\bar{B}} = 0$, so the second term in the right side of (5.6) vanishes. The third term in (5.6) vanishes because $\theta^a = 0$, and similarly $i\delta_c^a \phi_A \wedge \theta^A = i\delta_c^a \phi_\mu \wedge \theta^\mu$. The last two terms in the right side of (5.6) vanish because we are working mod θ and because the pseudoconformal curvature of the quadric is zero. Hence we have

$$d\hat{\phi}_c^a \equiv \hat{\phi}_c^\rho \wedge \hat{\phi}_\rho^a - i\delta_c^a \hat{\phi}_\mu \wedge \theta^\mu.$$

Now using Proposition 3.2.1, and equation (2.7), $\omega_{B\bar{A}} + \omega_{\bar{A}B} = 0$, we have

$$\begin{aligned} \hat{\phi}_c^\rho &\equiv \omega_c^\rho &\equiv \omega_{c\bar{A}} g^{\rho\bar{A}} \\ &&\equiv -\omega_{\bar{A}c} g^{\rho\bar{A}} \\ &&\equiv -\overline{\omega_A^B g_{B\bar{c}} g^{A\bar{\rho}}} \\ &&\equiv -\overline{\omega_\mu^d g^{\rho\bar{\mu}} g_{c\bar{d}}}. \end{aligned}$$

Hence the coefficient of $\theta^\alpha \wedge \theta^{\bar{\beta}}$ from $\hat{\phi}_c^\rho \wedge \hat{\phi}_\rho^a$ is $\omega^{\bar{\rho}a}_\alpha \omega_{\bar{\rho}c\bar{\beta}}$.

We next examine $i\delta_c^a \hat{\phi}_\mu \wedge \theta^\mu$ and work mod θ . We notice that

$$i\delta_c^a \hat{\phi}_\mu \wedge \theta^\mu = -i\delta_c^a \theta^\mu \wedge (g_{\mu\bar{A}} \hat{\phi}^{\bar{A}}) = -i\delta_c^a \theta^\mu \wedge (g_{\mu\bar{\sigma}} \hat{\phi}^{\bar{\sigma}})$$

due to the form of the matrix (g) . We substitute for $\hat{\phi}^{\bar{\sigma}}$ using equation (5.1) and use Proposition 3.2.1 to obtain

$$i\delta_c^a \theta^\mu \wedge (g_{\mu\bar{\sigma}} (\phi^{\bar{\sigma}} + C_{\bar{\rho}}^{\bar{\sigma}} \theta^{\bar{\rho}} + F^{\bar{\sigma}} \theta)) \equiv i\delta_c^a \theta^\mu \wedge (g_{\mu\bar{\sigma}} (\tau^{\bar{\sigma}} + (D_{\bar{\rho}}^{\bar{\sigma}} + C_{\bar{\rho}}^{\bar{\sigma}}) \theta^{\bar{\rho}})).$$

We notice that by equation (2.5), $\tau^{\bar{\sigma}}$ will be a combination of only forms like θ^γ , so we may ignore it when searching for coefficients of $\theta^\alpha \wedge \theta^{\bar{\beta}}$. We also note that $\hat{D}_{\bar{\beta}}^{\bar{\alpha}} = C_{\bar{\beta}}^{\bar{\alpha}} + D_{\bar{\beta}}^{\bar{\alpha}}$ by (5.1), so after lowering an index, we find the coefficient of $i\delta_c^a \hat{\phi}_\mu \wedge \theta^\mu$ in front of $\theta^\alpha \wedge \theta^{\bar{\beta}}$ is exactly $(\omega_{\bar{\rho}c\bar{\beta}}^{\bar{\rho}a} + i\delta_c^a \hat{D}_{\bar{\beta}\alpha})$, as desired. \square

5.3 Proof of Theorem 5.0.1

We first prove by induction that for all $j, k \geq 2$, we have

$$g_{a\bar{b}} \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_j}^a \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}} \equiv \mathring{g}_{a\bar{b}} \mathring{\omega}_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_j}^a \mathring{\omega}_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}}, \quad (5.7)$$

where equivalence here and in the rest of the proof means that the sides of the equation differ by a conformally flat tensor. We then show that such conformal equivalence is in fact equality and apply Lemma 5.1.3. We induct on the sum of the indices. By subtracting the Gauss equation for $\omega_{\gamma_1 \gamma_2}^a$ from the corresponding one for $\mathring{\omega}_{\gamma_1 \gamma_2}^a$, we obtain

$$g_{a\bar{b}} \omega_{\gamma_1 \gamma_2}^a \omega_{\bar{\alpha}_1 \bar{\alpha}_2}^{\bar{b}} \equiv \mathring{g}_{a\bar{b}} \mathring{\omega}_{\gamma_1 \gamma_2}^a \mathring{\omega}_{\bar{\alpha}_1 \bar{\alpha}_2}^{\bar{b}}, \quad (5.8)$$

since the pseudoconformal curvature tensor $S_{\gamma_1 \bar{\alpha}_1 \gamma_2 \bar{\alpha}_2}$ is computed using the same coframe (θ, θ^α) . This establishes the base step of the induction. We now assume equation (5.7) with $j+k \leq p$, and we wish to show the same where $j+k = p+1$. We will demonstrate the case where k increases by 1. The case where j increases is similar and left to the reader. We differentiate both sides of (5.7) in the $\theta^{\bar{\alpha}_{k+1}}$ direction, and noting that covariant derivatives of conformally flat tensors are conformally flat, obtain

$$\begin{aligned} g_{a\bar{b}} \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_j \bar{\alpha}_{k+1}}^a \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}} + g_{a\bar{b}} \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_j}^a \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k \bar{\alpha}_{k+1}}^{\bar{b}} \equiv \\ \mathring{g}_{a\bar{b}} \mathring{\omega}_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_j \bar{\alpha}_{k+1}}^a \mathring{\omega}_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}} + \mathring{g}_{a\bar{b}} \mathring{\omega}_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_j}^a \mathring{\omega}_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k \bar{\alpha}_{k+1}}^{\bar{b}}. \end{aligned}$$

The next lemma shows the equivalence of the first terms on each side of the above equation. We then subtract to finish the induction and demonstrate equation (5.7) for all $j, k \geq 2$.

Lemma 5.3.1. *With the same setup as above, we have*

$$g_{a\bar{b}} \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_j \bar{\alpha}_{k+1}}^a \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}} \equiv \mathring{g}_{a\bar{b}} \mathring{\omega}_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_j \bar{\alpha}_{k+1}}^a \mathring{\omega}_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}}$$

Proof. We induct on s , the position of the index $\bar{\alpha}_{k+1}$. When $s = 3$, we need

$$g_{a\bar{b}} \omega_{\gamma_1 \gamma_2; \bar{\alpha}_{k+1} \gamma_3 \dots \gamma_j}^a \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}} \equiv \mathring{g}_{a\bar{b}} \mathring{\omega}_{\gamma_1 \gamma_2; \bar{\alpha}_{k+1} \gamma_3 \dots \gamma_j}^a \mathring{\omega}_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}}.$$

This follows immediately from Lemma 5.1.2, which implies that both sides are conformally flat.

We assume that the desired equivalence holds for $s = r$, where $r \leq j$, that is,

$$g_{a\bar{b}} \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_r \bar{\alpha}_{k+1} \gamma_{r+1} \dots \gamma_j}^a \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}} \equiv \mathring{g}_{a\bar{b}} \mathring{\omega}_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_r \bar{\alpha}_{k+1} \gamma_{r+1} \dots \gamma_j}^a \mathring{\omega}_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}}, \quad (5.9)$$

and we would like to show the same when $s = r + 1$:

$$g_{a\bar{b}} \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_r \bar{\alpha}_{k+1} \gamma_{r+1} \dots \gamma_j}^a \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}} \equiv \mathring{g}_{a\bar{b}} \mathring{\omega}_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_r \bar{\alpha}_{k+1} \gamma_{r+1} \dots \gamma_j}^a \mathring{\omega}_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}}. \quad (5.10)$$

By Lemma 5.2.1, we have

$$\begin{aligned} \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_r \bar{\alpha}_{k+1}}^a &\equiv \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \bar{\alpha}_{k+1} \gamma_r}^a + \sum_{q=1}^{r-1} d_{\gamma_r \bar{\alpha}_{k+1}} (\omega_{\gamma_q}^\mu) \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_{q-1} \mu \gamma_{q+1} \dots \gamma_{r-1}}^a \\ &\quad - (g^{\sigma\bar{\rho}} g_{c\bar{d}} \omega_{\sigma}^a \omega_{\bar{\rho}}^{\bar{d}} \bar{\alpha}_{k+1}) \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_{r-1}}^c - i(\hat{D}_{\bar{\alpha}_{k+1} \gamma_r}) \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_{r-1}}^a \end{aligned} \quad (5.11)$$

We take covariant derivatives of both sides of (5.11) in the $\theta^{\gamma_{r+1}}, \dots, \theta^{\gamma_j}$ directions successively, multiply by $g_{a\bar{b}} \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}}$, and analyze each term on the right side of the resulting equation. We will show that by (5.9), and equation (5.7) with $j + k \leq p$ (the induction hypotheses in the proof of Lemma 5.3.1 and the proof of Theorem 5.0.1 respectively), each such term must be conformally equivalent to the corresponding term with the ring superscript. This will demonstrate (5.10) and hence conclude the proof of Lemma 5.3.1. This is because we may also apply Lemma 5.2.1 to $\mathring{\omega}_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_r \bar{\alpha}_{k+1}}^a$, take covariant derivatives in the $\theta^{\gamma_{r+1}}, \dots, \theta^{\gamma_j}$ directions, and multiply by $\mathring{g}_{a\bar{b}} \mathring{\omega}_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}}$.

After taking covariant derivatives and multiplying by $g_{a\bar{b}} \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}}$, the first term on the right side of (5.11) will be $g_{a\bar{b}} \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_{r-1} \bar{\alpha}_{k+1} \gamma_r \dots \gamma_j}^a \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}}$ which is conformally equivalent to the same term with the ring superscript by (5.9). We also notice that after taking covariant derivatives and multiplying by $g_{a\bar{b}} \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}}$, the second term on the right side of equation (5.11) yields many terms, each of which is a product of covariant derivatives of $d_{\gamma_r \bar{\alpha}_{k+1}} (\omega_{\gamma_q}^\mu)$, covariant derivatives of $\omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_{q-1} \mu \gamma_{q+1} \dots \gamma_{r-1}}^a$, and $g_{a\bar{b}} \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}}$. We notice that expressions of the form $d_{\gamma_r \bar{\alpha}_{k+1}} (\omega_{\gamma_q}^\mu)$ are intrinsic to the manifold M and thus all covariant derivatives will be the same as those with

the ring superscript. Also, covariant derivatives of $\omega_{\gamma_1 \gamma_2 \gamma_3 \dots \gamma_{q-1} \mu \gamma_{q+1} \dots \gamma_{r-1}}^a$ multiplied by $g_{a\bar{b}} \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}}$ will be the same as those with the ring superscript by (5.7), since $r+k \leq j+k \leq p$.

The third term on the right side of equation (5.11) can be written as

$$-(g^{\sigma\bar{p}} \omega_{\sigma \gamma_r}^a)(g_{c\bar{d}} \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_{r-1}}^c \omega_{\bar{p} \bar{\alpha}_{k+1}}^{\bar{d}}).$$

We observe that $(g_{c\bar{d}} \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_{r-1}}^c \omega_{\bar{p} \bar{\alpha}_{k+1}}^{\bar{d}})$ is conformally equivalent to the same with the ring superscript by (5.7), and hence covariant derivatives will be also. Also, taking covariant derivatives of the term $g^{\sigma\bar{p}} \omega_{\sigma \gamma_r}^a$ and multiplying by $g_{a\bar{b}} \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}}$ will yield terms conformally equivalent to those with the ring superscript, again by (5.7) and because $g^{\sigma\bar{p}}$ is intrinsic to M .

In the last term on the right side of equation (5.11), we first show that $\hat{D}_{\bar{\alpha}_{k+1} \gamma_r}$ is conformally equivalent to the same with the ring superscript. Observe that by equation (5.2), we have

$$C_{\alpha\bar{\beta}} = \frac{i}{n+2} [\omega_{\mu \alpha}^a \omega_{a\bar{\beta}}^{\mu} - \frac{g_{\alpha\bar{\beta}}}{2(n+1)} \omega_{\mu \nu}^a \omega_a^{\mu \nu}].$$

Here we have used the vanishing of the pseudoconformal curvature of the target hyperquadric. We may write this as

$$C_{\alpha\bar{\beta}} = \frac{i}{n+2} [g^{\mu\bar{\nu}} (g_{a\bar{b}} \omega_{\mu \alpha}^a \omega_{\bar{\nu} \bar{\beta}}^{\bar{b}}) - \frac{g_{\alpha\bar{\beta}} g^{\mu\bar{\sigma}} g^{\nu\bar{\beta}}}{2(n+1)} (g_{a\bar{b}} \omega_{\mu \nu}^a \omega_{\bar{\sigma} \bar{\beta}}^{\bar{b}})].$$

Equation (5.8) implies conformal equivalence of both terms of the form $(g_{a\bar{b}} \omega_{\mu \alpha}^a \omega_{\bar{\nu} \bar{\beta}}^{\bar{b}})$ with the corresponding terms with superscripts. Since $\hat{D}_{\bar{\beta}}^{\bar{\alpha}} = C_{\bar{\beta}}^{\bar{\alpha}} + D_{\bar{\beta}}^{\bar{\alpha}}$ (see (5.1)), and the term $D_{\bar{\beta}}^{\bar{\alpha}}$ is intrinsic to M , we have that $\hat{D}_{\bar{\alpha}_{k+1} \gamma_r}$ is conformally equivalent to its counterpart with the ring superscript.

After taking covariant derivatives and multiplying by $g_{a\bar{b}} \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}}$ in the last term on the right side of equation (5.11), every resulting term will be a product of derivatives of $\hat{D}_{\bar{\alpha}_{k+1} \gamma_r}$, derivatives of $\omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_{r-1}}^a$, and $g_{a\bar{b}} \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}}$. The derivatives of $\hat{D}_{\bar{\alpha}_{k+1} \gamma_r}$ will be conformally equivalent to the same with the ring superscript, as explained above, and the remaining terms will be conformally equivalent to their counterparts with the ring superscript by (5.7). This concludes the proof of Lemma 5.3.1

□

We now return to the proof of Theorem 5.0.1. We have shown that

$$g_{a\bar{b}} \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_j}^a \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}} \equiv \mathring{g}_{a\bar{b}} \mathring{\omega}_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_j}^a \mathring{\omega}_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}}$$

where the equivalence is modulo a conformally flat tensor. Our next step is to show that this equivalence is in fact equality. We will demonstrate this equality in the case where $l = n/2$ or f is side-preserving. To do this, we make use of Lemmas 2.5.1 and 2.5.2. We first show equality in the case where $j = k$ using Lemma 2.5.2. At the end of the proof we mention the side-reversing case.

First, suppose $(N_0 - n) + (l' - l) < l$ and consider the following expression

$$g_{a\bar{b}} \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_k}^a \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}} - \mathring{g}_{a\bar{b}} \mathring{\omega}_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_k}^a \mathring{\omega}_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}} \equiv 0. \quad (5.12)$$

Let $\zeta := (\zeta^1, \dots, \zeta^n)$, multiply equation (5.12) by $\zeta^{\gamma_1} \zeta^{\bar{\alpha}_1} \dots \zeta^{\gamma_k} \zeta^{\bar{\alpha}_k}$ and sum. Since the right side of (5.12) is conformally flat, we have

$$-\sum_{a=1}^{l'-l} |\omega^a(\zeta)|^2 - \sum_{b=1}^{N_0-n} |\mathring{\omega}^b(\zeta)|^2 + \sum_{a=l'-l+1}^{N-n} |\omega^a(\zeta)|^2 = A(\zeta, \bar{\zeta}) \left(-\sum_{i=1}^l |\zeta^i|^2 + \sum_{j=l+1}^n |\zeta^j|^2 \right),$$

where $\omega^a(\zeta) = \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_k}^a \zeta^{\gamma_1} \dots \zeta^{\gamma_k}$, $\mathring{\omega}^b(\zeta) = \mathring{\omega}_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_k}^b \zeta^{\gamma_1} \dots \zeta^{\gamma_k}$, and $A(\zeta, \bar{\zeta})$ is a polynomial in ζ and $\bar{\zeta}$. Since we have $(N_0 - n) + (l' - l) < l$, Lemma 2.5.2 implies that $A(\zeta, \bar{\zeta})$ is identically zero, so we have the desired equality, which we may rewrite as

$$\sum_{a=l'-l+1}^{N-n} |\omega^a(\zeta)|^2 = \sum_{a=1}^{l'-l} |\omega^a(\zeta)|^2 + \sum_{b=1}^{N_0-n} |\mathring{\omega}^b(\zeta)|^2. \quad (5.13)$$

Now suppose that $(N - l') - (n - l) < l$. We consider the expression

$$\mathring{g}_{a\bar{b}} \mathring{\omega}_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_k}^a \mathring{\omega}_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}} - g_{a\bar{b}} \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_k}^a \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}} \equiv 0.$$

By noticing that $(N - l') - (n - l) = (N - n) - (l' - l)$ and performing a similar argument we use Lemma 2.5.2 to obtain the desired equality. The details are left to the reader.

Now we will show that the conformal equivalence is actually an equality in the expression

$$g_{a\bar{b}} \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_j}^a \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}} \equiv \mathring{g}_{a\bar{b}} \mathring{\omega}_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_j}^a \mathring{\omega}_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}} \quad (5.14)$$

where without loss of generality, we assume $j > k$. We first assume $(N_0 - n) + (l' - l) < l$ and rewrite equation (5.14) as

$$\begin{aligned}
& - \sum_{a=1}^{l'-l} \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_j}^a \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{a}} - \sum_{b=1}^{N_0-n} \hat{\omega}_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_j}^b \hat{\omega}_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}} \\
& \quad + \sum_{c=l'-l+1}^{N-n} \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_j}^c \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{c}} \equiv 0. \quad (5.15)
\end{aligned}$$

We apply a lemma of D'Angelo (see [DA93], chapter 5) to equation (5.13) to obtain the existence of a unitary matrix U such that

$$U \begin{pmatrix} \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{1}} \\ \vdots \\ \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\overline{l'-l}} \\ \hat{\omega}_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{1}} \\ \vdots \\ \hat{\omega}_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\overline{N_0-n}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\overline{l'-l+1}} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\overline{N-n}} \end{pmatrix}$$

Note that we are working at a fixed point here. This implies the existence of constants \bar{A}_r^c and \bar{B}_s^c , with $1 \leq r \leq l' - l$ and $1 \leq s \leq N_0 - n$ such that

$$\omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{c}} = \bar{A}_r^c \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{r}} + \bar{B}_s^c \hat{\omega}_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{s}},$$

where $l' - l + 1 \leq c \leq N - n$, and we are using the summation convention for the indices r and s .

We substitute the above into equation (5.15) to obtain

$$\begin{aligned}
& - \sum_{a=1}^{l'-l} \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_j}^a \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{a}} - \sum_{b=1}^{N_0-n} \hat{\omega}_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_j}^b \hat{\omega}_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{b}} \\
& \quad + \sum_{c=l'-l+1}^{N-n} \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_j}^c (\bar{A}_r^c \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{r}} + \bar{B}_s^c \hat{\omega}_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{s}}) \equiv 0.
\end{aligned}$$

We regroup the terms in this expression, which yields

$$\begin{aligned} \sum_{r=1}^{l'-1} \left(\sum_{c=l'-l+1}^{N-n} \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_j}^c \bar{A}_r^c - \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_j}^r \right) \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{r}} \\ + \sum_{s=1}^{N_0-n} \left(\sum_{c=l'-l+1}^{N-n} \omega_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_j}^c \bar{B}_s^c - \dot{\omega}_{\gamma_1 \gamma_2; \gamma_3 \dots \gamma_j}^s \right) \dot{\omega}_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{s}} \equiv 0, \end{aligned}$$

where we are not using the summation convention for the indices r and s . Since the number of terms in the sum on the left side in the preceding equation is strictly less than n , we use Lemma 2.5.1 in the same way that we used Lemma 2.5.2 previously to conclude that the conformal equivalence is in fact an equality. We then recombine all terms to get the desired equality. In the case where $(N - l') - (n - l) < l$, we apply the lemma of D'Angelo as above to obtain constants \bar{A}_c^r and \bar{B}_c^s , such that

$$\omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{r}} = \bar{A}_c^r \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{c}}, \quad \text{and} \quad \dot{\omega}_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{s}} = \bar{B}_c^s \omega_{\bar{\alpha}_1 \bar{\alpha}_2; \bar{\alpha}_3 \dots \bar{\alpha}_k}^{\bar{c}},$$

where $1 \leq r \leq l' - l$, $1 \leq s \leq N_0 - n$, $l' - l + 1 \leq c \leq N - n$ and we are using the summation convention on the indices r and s . We then substitute into (5.15) as before to obtain the desired result. The details of this are left to the reader.

We embed the vectors representing the second fundamental form of f_0 and its derivatives into \mathbb{C}^{N-n} by appending the appropriate number of zeros. Thus we have shown that all inner products of derivatives of the second fundamental form of f with respect to $g_{a\bar{b}}$ are equal to the corresponding inner products of derivatives of the second fundamental form of f_0 with respect to $\dot{g}_{a\bar{b}}$. Lemma 5.1.3 gives the desired inequality relating the dimension of E_k and \dot{E}_k .

In the side reversing case, the argument is similar except that we need only consider the analogue of the negative of equation (5.12). This is because $\min(N - l' - l, l' - (n - l)) = l' - (n - l)$. We leave the details to the reader.

5.4 Proof of Theorem 1.2.2

We may now prove Theorem 1.2.2. We use the notation of Theorem 5.0.1.

Proof of Theorem 1.2.2. Recall that $\mu(M)$ denotes the CR complexity of M as defined in (1.1). Let $N_0 = n + \mu(M)$ and $f_0: M \rightarrow Q_l^{N_0}$ a CR transversal CR map (whose existence

is guaranteed by the definition of $\mu(M)$. If $l = n/2$, or f is side preserving, we notice that $l' \geq l$ and $N - l' \geq n - l$ by Proposition 2.3.1. Next, we apply Theorem 5.0.1. Since $\dim E_k \leq (N_0 - n) + \min(l' - l, (N - n) - (l' - l))$ for all k , the degeneracy of f is at least $(N - n) - (N_0 - n) - \min(l' - l, (N - n) - (l' - l))$, so if s denotes the degeneracy of f at a generic point on M where f is constantly (k, s) -degenerate for some k , we have $s \geq (N - N_0) - \min(l' - l, (N - n) - (l' - l))$. Since $(N - n) - s \leq (N_0 - n) + (l' - l) < n$, we may apply Theorem 1.2.1 to obtain the desired conclusion in Theorem 1.2.2 (near a generic point). We note here that it suffices to prove that the image of f is contained in the complex plane P in the neighborhood of some point on M to obtain the full conclusion.

If f is side reversing, we notice that $N - l' \geq l$ and $l' \geq n - l$ by Proposition 2.3.1. We apply Theorem 5.0.1 again to see that the degeneracy of f is at least $(N - n) - (N_0 - n) - (l' - (n - l))$. Denoting the degeneracy by s again, we have $s \geq (N - l' - l) + (n - N_0)$. Since $(N - n) - s \leq (N_0 - n) + l' - (n - l) < n$, we may apply Theorem 1.2.1 to obtain the desired conclusion as above. \square

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